ETH Zürich, D-INFK
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Diskrete Mathematik

Exercise 14

14.1 Prenex Normal Form (⋆)

For each of the following formulas, find an equivalent formula in the prenex normal form.

i) $\big(\forall x\ P(x)\big) \rightarrow Q(x)$

ii) $\forall z \exists y\ \big(P(x, g(y), z) \vee \neg\forall x Q(x)\big)\ \wedge\ \neg\forall z \exists x\ \neg R(f(x, z), z)$

14.2 The Barber of Zürich (⋆)

Use Theorem 6.13 to show that there does not exist in Zürich a barber who shaves all those and exactly those who do not shave themselves.

14.3 The Exercise 14 (⋆ ⋆ ⋆)

Prove the following statement about the students who attend the Discrete Mathematics lecture:

> "There exists a student, such that if this student solves Exercise 14, then all students solve Exercise 14."

To this end, proceed in following steps:

a) Let U be the set of all students who attend the Discrete Mathematics lecture. Let P be the predicate defined as $P(s) = 1$ if and only if the student s solves Exercise 14. Describe the above statement by a formula F that uses P.

b) Show that F is a tautology (that is, show that it is true for any U and P).

c) Find a different (interesting) interpretation for F, which defines U and P.

14.4 Formulas and Statements (⋆ ⋆)

For each of the following expressions, determine whether it is syntactically correct, and, if so, whether it is a formula or a statement about formulas.[1] If an expression is a statement, decide whether it is true or false (each time justify your answer).

a) $\forall x\ \exists y\ \big(P(z) \leftrightarrow Q(f(f(x, z), y))\big)$

b) $\big(\forall x\ P(x)\big) \models P(x)$

[1] Whenever parentheses are not necessary, they can be omitted. Parentheses do not influence correctness.

c) $(P(x) \models P(x)) \equiv Q(x)$

d) $\{P(x), P(f(a))\} \models P(a)$

14.5 Calculi

a) ($\star$) Decide which of the following rules are correct (justify your answers):

$$\{F\} \vdash_{R_1} F \vee G \qquad \{F \wedge G\} \vdash_{R_2} F \qquad \{\neg(F \wedge G)\} \vdash_{R_3} \neg F \wedge \neg G$$
$$\{F, F \to G\} \vdash_{R_4} G \qquad \{F \to G\} \vdash_{R_5} \neg F \to \neg G \qquad \{F, G\} \vdash_{R_6} F \wedge G$$

b) ($\star\,\star$) Let K be the calculus, consisting of those of the rules in Subtask a), which are correct. Using K, derive formally the formula $A \wedge B \wedge C \wedge D$ from the following set of formulas:

$$\{(D \wedge A) \to C,\ A \wedge B,\ B \wedge A,\ (B \vee C) \to D\}$$

c) ($\star\,\star$) Is $K' = \{R_2, R_4\}$ complete? Justify your answer.

d) ($\star\,\star$) Give an example of a calculus, which is complete but not sound.

14.6 Resolution

a) ($\star$) Prove the following statements using the resolution calculus.

i) $F = (A \vee B) \wedge (\neg E) \wedge (\neg B \vee D) \wedge (\neg D \vee E) \wedge (\neg A \vee B)$ is not satisfiable.

ii) $G = (\neg B \wedge \neg C \wedge D) \vee (\neg B \wedge \neg D) \vee (C \wedge D) \vee B$ is a tautology.

iii) $H = A \wedge C$ is a logical consequence of $M = \{A \to C, B \to A, A \vee B\}$.

b) ($\star\,\star\,\star$) Intuitively, the goal is to show that from a finite set of finite clauses, after a finite number of applications of derivation rules, no new clauses can be derived. More specifically, let $\mathcal{K}$ be a finite set of finite clauses and let $\mathcal{K}_0, \mathcal{K}_1, \ldots$ be a sequence of applications of derivation rules, such that $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_i = \mathcal{K}_{i-1} \cup \{K\}$ for all $i > 0$, where $\{K', K''\} \vdash_{\mathsf{res}} K$ for some $K', K'' \in \mathcal{K}_{i-1}$. Show that there exists an n such that $\mathcal{K}_m = \mathcal{K}_n$ for all $m > n$.

c) ($\star\,\star\,\star$) Show that the statement from Subtask b) is no longer true for an infinite set $\mathcal{K}$ of finite clauses. More precisely, let $\mathcal{K} = \{\{A_j, \neg A_{j+1}\} \mid j \in \mathbb{N}\}$. Show that there exists an infinite sequence $\mathcal{K}_0, \mathcal{K}_1, \ldots$, such that $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_i = \mathcal{K}_{i-1} \cup \{K\}$ for all $i > 0$, where $\{K', K''\} \vdash_{\mathsf{res}} K$ for some $K', K'' \in \mathcal{K}_{i-1}$, and for all $i > 0$, $\mathcal{K}_i \neq \mathcal{K}_{i-1}$.