ETH Zürich, D-INFK
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Diskrete Mathematik

Exercise 5

Exercise 5.1 gives **bonus points**, which can increase the final grade. The solution to this exercise must be your own work. You may not share your solutions with anyone else. See also the note on dishonest behavior on the course website: `https://crypto.ethz.ch/teaching/DM20/`.

5.1 A Property of Any Two Sets ($\star\star$) *(8 Points)*

Prove or disprove: for any two sets A and B there exists a set C such that

$$A = (B \setminus C) \cup (C \setminus B).$$

5.2 Relating Two Power Sets ($\star\star$)

Prove or disprove each of the following statements.

a) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ for any sets A and B.

b) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ for any sets A and B.

c) $A \subseteq B \Longleftrightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$ for any sets A and B.

5.3 Family Relations ($\star$)

Consider the set of all people (both living and already dead) and different family relations on this set: *id* (identity), *if* (is the father of), *im* (is the mother of), *ip* (is a parent of) and *ic* (is a child of).

a) Express each of the following relations, using the relations above.

i) x *iggf* $y \overset{\text{def}}{\Longleftrightarrow} x$ is a great-grandfather of y

ii) x *ihs* $y \overset{\text{def}}{\Longleftrightarrow} x$ is a half-sibling of y (i.e., x and y share exactly one parent)

iii) x *ico* $y \overset{\text{def}}{\Longleftrightarrow} x$ is a cousin of y (i.e., x and y are not siblings but have a common grandparent)

b) What is the relationship between the relations *ic* $\circ$ *ic* $\circ$ *ip* $\circ$ *ip* and *ic* $\circ$ *ip* $\circ$ *ic* $\circ$ *ip*? Are they the same relation or is one of them a subset of the other?

5.4 Computing Representations of Relations ($\star$)

For the relation $\rho = \{(1,4),(2,1),(2,3),(4,2)\}$ on the set $\{1,2,3,4\}$, determine the relations ρ^3 and ρ^*. Describe ρ^3 using the set representation, and ρ^* using matrix representation.

5.5 Operations on Relations ($\star\star$)

Let us consider the relations $<, \mid$ and $\equiv_2$ on the set of positive natural numbers $\mathbb{N} \setminus \{0\}$. For each of the following relations on $\mathbb{N} \setminus \{0\}$, decide whether it is reflexive, symmetric or transitive. Justify your answers.

a) $< \circ \mid$

b) $\mid \cup \equiv_2$

c) $\mid \cup \mid^{-1}$

5.6 A False Proof ($\star$)

Consider a non-empty set A and a symmetric and transitive relation $\rho \neq \varnothing$ on A.

a) The following proof shows that ρ is always reflexive. Find the mistake in this proof.

Proof: We show that ρ is reflexive, that is that for any x, we have $x \; \rho \; x$. Let $x \in A$. Further, let $y \in A$ be such that $x \; \rho \; y$. Since ρ is symmetric, it follows that $y \; \rho \; x$. Now we have $x \; \rho \; y$ and $y \; \rho \; x$. Hence, by the transitivity of ρ, it follows that $x \; \rho \; x$.

b) Show that the above statement is indeed false, that is, prove that ρ is not always reflexive.

5.7 Properties of Relations ($\star\star$)

Prove or disprove each of the following statements.

a) If a relation σ on a set A is not reflexive, then the relation σ^2 is also not reflexive.

b) If relations σ and ρ on a set A are antisymmetric, then so is the relation $\sigma \cap \rho$.

Due by 20. October 2020.
Exercise 5.1 is graded.