ETH Zürich, D-INFK
HS 2020, 2. October 2020

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Diskrete Mathematik

Exercise 4

Exercises 4.5 and 4.6 give **bonus points**, which can increase the final grade. The solution to this exercise must be your own work. You may not share your solutions with anyone else. See also the note on dishonest behavior on the course website: `https://crypto.ethz.ch/teaching/DM20/`.

Part 1: Proof Patterns

4.1 Indirect Proof of an Implication (2.6.3)

Prove indirectly that for all natural numbers $n > 0$, we have:

a) ($\star$) If n^2 is odd, then n is also odd.

b) ($\star\star$) If $42^n - 1$ is a prime, then n is odd.

4.2 Case Distinction (2.6.5)

Prove by case distinction that:

a) ($\star$) $n^3 + 2n + 6$ is divisible by 3 for all natural numbers $n \geq 0$.

b) ($\star\star$) If p and $p^2 + 2$ are primes, then $p^3 + 2$ is also a prime.

4.3 Proof by Contradiction (2.6.6)

a) ($\star\star$) Show by contradiction that the sum of a rational number and an irrational number is irrational.

Hint: Use the fact that the difference of two rational numbers is rational.

b) ($\star\star\star$) Show that the number $2^{\frac{1}{n}}$ is irrational for $n > 2$, by reaching a contradiction with Fermat's Last Theorem.

Hint: Fermat's Last Theorem states that no positive integers a, b, c satisfy the equation $a^n + b^n = c^n$ for $n > 2$.

4.4 Pigeonhole Principle (2.6.8)

a) ($\star\star$) Given are five points on a sphere. Show that there exists a closed hemisphere (i.e., a hemisphere that includes its boundary) which contains four of these points.

b) ($\star\star\star$) On the 1^{st} of November, a monkey bought 45 bananas. On each of the following 30 days it ate a certain number of its bananas. It had to eat at least one banana every day, in order not to starve. Prove that there must exist a period of consecutive days in which the monkey ate exactly 14 bananas.

4.5 On the Soundness of a New Proof Pattern ($\star$) *(6 Points)*

Consider the following proof pattern. We prove a statement S in four steps:

1) Find two suitable mathematical statements T and U.
2) Prove T, assuming S is false.
3) Prove U, assuming S is false.
4) Prove that not both T and U are true.

We want to use propositional logic to reason about the soundness of this proof pattern.

a) Phrase a statement of the form $F \models G$ (for suitably chosen propositional formulas F and G) that explains the soundness of the proof pattern.

b) Prove or disprove the proof pattern's soundness by proving either $F \models G$ or $F \not\models G$.

Part 2: Set Theory

4.6 Element or Subset ($\star$) *(2 Points)*

For each of the following choices of sets A and B, decide which of the statements $A \in B$ and $A \subseteq B$ are true.

i) $A = \{1, 0, \{0\}, 1\}, B = \{\{0, 1, 0, \{0, 0\}\}, 1, 10, 0\}$ **ii)** $A = \varnothing, B = \{\{\varnothing\}, \{\varnothing, \varnothing\}, \varnothing\}$

iii) $A = \{\{0\}, 0, \{0\}, \{\{0\}\}\}, B = \{0, \varnothing, \{\{0\}\}, \{0\}\}$ **iv)** $A = \{\varnothing\}, B = \{\varnothing, \{\varnothing, \varnothing, \varnothing\}\}$

4.7 Operations on Sets ($\star$)

In each of the following cases, give a set A such that

a) there exists an $x \in A$ such that $x \subseteq A$.

b) $A \not\subseteq \mathcal{P}(A)$ and there exists an $x \in A$ such that $x \subseteq \mathcal{P}(A)$.

c) $A \subseteq \mathcal{P}(A)$ and for all $x \in A$ it holds that $x \not\subseteq \mathcal{P}(A)$.

4.8 Cardinality ($\star$)

Let $A = \{\varnothing, \{\varnothing\}, \{\varnothing\}\}$ and $B = \{A, \{\varnothing\}, \{\{\varnothing\}\}\}$. Specify each of the following sets (by listing all its elements) and give its cardinality.

i) $A \cup B$ **ii)** $A \cap B$ **iii)** $\varnothing \times A$

iv) $\{0\} \times \{3, 1\}$ **v)** $\{\{1, 2\}\} \times \{3\}$ **vi)** $\mathcal{P}(\{\varnothing\})$

Due by 13. October 2020.
Exercises 4.5 and 4.6 are graded.