ETH Zürich, D-INFK
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Diskrete Mathematik

Exercise 2

Exercise 2.3 gives **bonus points**, which can increase the final grade. The solution to this exercise must be your own work. You may not share your solutions with anyone else. See also the note on dishonest behavior on the course website: `https://crypto.ethz.ch/teaching/DM20/`.

2.1 Interpreting Propositional Formulas in Natural Language

Let A be the proposition "The monkey sits on the palm tree." and let B be the proposition "The monkey has a banana.".

a) ($\star$) How would you interpret the following formulas?

i) $F_1 = B \land \neg A$ **ii)** $F_2 = (A \land B) \lor (\neg A \land \neg B)$

b) ($\star$) Using only the propositions A, B and logical operators, write down formulas corresponding to the following sentences:

i) F_3: "The monkey neither sits on the palm tree nor has a banana."

ii) F_4: "The monkey either has a banana or sits on the palm tree, but not both."

c) ($\star\star$) For both formulas F_3, F_4, write down their negations both as sentences and formally as formulas.

2.2 Logical Equivalence via Function Tables

a) ($\star$) Compute the function table for the following formula:

$$(B \to C) \to \big(\neg(A \to C) \land \neg(A \lor B)\big).$$

b) ($\star\star$) Give another formula that is equivalent to the formula from Subtask a), but in which each of the propositional symbols appears at most once.

2.3 Proving Logical Equivalence using Equivalence Transformations ($\star\star$) *(8 Points)*

Consider the formulas

$$\begin{aligned} F &= (C \land A) \lor ((B \to A) \land \neg C) \quad \text{and} \\ G &= A \lor \neg(B \lor C). \end{aligned}$$

Prove the statement $F \equiv G$ by using equivalence transformations.

Expectation. Your proof has to be in the form of a sequence of *at most* 16 steps, where each step consists of applying the definition of $\rightarrow$ (that is $F \rightarrow G \equiv \neg F \vee G$), one of the rules given in Lemma 2.1 of the lecture notes[1], or one of the following rules: $F \wedge \neg F \equiv \bot$, $F \wedge \bot \equiv \bot$, $F \vee \bot \equiv F$, $F \vee \neg F \equiv \top$, $F \wedge \top \equiv F$, and $F \vee \top \equiv \top$. For this exercise, associativity is to be applied as in Lemma 2.1 *3)*. Each step of your proof has to apply a *single* rule *once* and state *which* rule was applied.

Verify your solution very carefully before handing it in.

2.4 Logical Consequence

Prove or disprove the following statements about formulas.

a) $(\star)$ $A \wedge (A \rightarrow B) \models B$

b) $(\star)$ $A \rightarrow B \models \neg A \rightarrow \neg B$

c) $(\star\star)$ $(A \rightarrow B) \wedge (B \rightarrow C) \models (A \rightarrow C)$

2.5 Satisfiability and Tautologies ($\star$)

For each of the formulas below, determine whether it is satisfiable or unsatisfiable and whether it is a tautology or not. Prove your answers.

a) $(A \vee B) \wedge \neg A$

b) $\big((A \rightarrow B) \wedge (B \rightarrow C)\big) \wedge \neg(A \rightarrow C)$

2.6 Knights and Knaves ($\star\star\star$)

We find ourselves on a strange island with only two types of inhabitants: knights and knaves. The knights always tell the truth, while the knaves always lie. From the outside, both groups look exactly the same and we cannot distinguish one from the other.

We have lost our way and come to a fork in the road. We know that one of the roads leads to a deadly jungle, while the other will take us to a friendly village. We see an islander standing at the fork. He is willing to answer only one question and his answer can only be "Yes" or "No". What question do we ask?

We want to use propositional logic to solve this problem. Let A be the proposition "The left road leads to the village." and let B be the proposition "The islander is a knight." We phrase our question as a formula F in A and B, asking the islander about the truth value of F. How do we choose F such that we are guaranteed to learn which road leads to the village?

Due by 29. September 2020.
Exercise 2.3 will be graded.

[1] Lemma 2.1 states rules involving propositional symbols, but you may apply those rules at the level of formulas (see Section 2.3.5 of the lecture notes).