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Prof. Ueli Maurer
David Lanzenberger

Diskrete Mathematik

Exercise 1

This exercise sheet contains some mathematical riddles and brain teasers, which are, in one way or another, connected to the topics of this lecture. The solutions will be discussed during the first exercise class. You are not expected to be able to fully solve all of the exercises.

1.1 The Punctured Chessboard ($\star$)

In the lecture (see also Example 1.1 in the lecture notes) we considered a $k \times k$ chessboard with one of the squares punctured. We also defined the predicate $P(k)$ to be equal 1 whenever the following statement is true:

> *No matter which of the squares is punctured, the remaining area of the chessboard (consisting of $k^2 - 1$ squares) can be covered completely with (non-overlapping) L-shaped pieces of paper consisting of three squares.*

In this exercise we consider the proof by case distinction that $P(7) = 1$.

a) What is the smallest number of cases one has to consider in the proof? (Consider symmetries of the chessboard.)

b) Carry out the proof for two of the cases.

1.2 A False Proof ($\star\star$)

Find the mistake in the following proof.

Claim: 1 is the largest natural number.

Proof:
n is the largest natural number

$\implies n^2 \leq n.$
$\implies n(n-1) = n^2 - n \leq 0$
$\implies 0 \leq n \leq 1$
$\implies n = 1.$

1.3 Hilbert's Hotel

David Hilbert [1] is the manager of a hotel with an infinite number of rooms, numbered by $1, 2, 3, \ldots$, all of which are occupied. Each guest is willing to change his or her room at

[1] David Hilbert (1862 – 1943) was a prominent mathematician, who in fact proposed a thought experiment, concerning a hotel with an infinite number of rooms.

most once.

a) ($\star\star$) Roger Federer comes to the hotel and asks whether there is a free room for him. Hilbert cannot turn away the distinguished guest and promises him a room. To make it possible, he tells some of the other guests to change their rooms. How can he do this, so that at the end every current guest still has a room?

b) ($\star\star\star$) One day, at the door of the hotel arrives a bus with an infinite number of passengers, numbered by $1, 2, 3, \ldots$, each of whom would like a single room. How can Hilbert accommodate all newly arrived guests, so that all current guests still have a place to stay?

c) ($\star\star\star\star$) An infinite number of buses, numbered by $1, 2, 3, \ldots$, each carrying an infinite number of passengers, arrives at the hotel. How can Hilbert deal with this situation?