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Diskrete Mathematik

Solution 13

13.1 Free Variables

i) $\forall x \forall y\ (P(x,y) \vee P(x,\underline{\mathbf{z}}))$

ii) $\big(\forall x\ (\exists x\ P(x) \wedge P(x)) \vee P(\underline{\mathbf{x}})\big)$
In the first occurence of $P(x)$, x is bound by $\exists x$ and in the second occurrence it is bound by $\forall x$.

iii) There are no free variables in this formula.

13.2 Interpretations

a) **i)** $\mathcal{A}$ is a model for F, because for all positive natural numbers x, y, z we have:

$$x \mid xy \;\;\wedge\;\; y \mid xy \;\;\wedge\;\; \big(y \nmid x \rightarrow yz \nmid x\big).$$

ii) $\mathcal{A}$ is not a model for F, because there exist positive natural numbers x, y, z, for which the following does not hold:

$$x \mid x^y \;\;\wedge\;\; y \mid x^y \;\;\wedge\;\; (y \nmid x \rightarrow y^z \nmid x).$$

The counterexample is $x = 2, y = 3$ (note that $y \nmid x^y$).

iii) $\mathcal{A}$ is a model for F, because for all subsets A, B, C of $\mathbb{N}$ we have:

$$A \cap B \subseteq A \;\;\wedge\;\; A \cap B \subseteq B \;\;\wedge\;\; (A \not\subseteq B \rightarrow A \not\subseteq B \cap C).$$

b) There are many correct solutions. Below we give an example.

i) The structure $\mathcal{A}$ that defines only the universe: $U^{\mathcal{A}} = \{0\}$.

ii) The structure $\mathcal{A}$ with $U^{\mathcal{A}} = \{0\}$ and $P^{\mathcal{A}}(0,0) = 0$. $\mathcal{A}$ is not a model, because $\forall x \exists y\ P(x,y)$ is false (since $P(x,y)$ is always false).

iii) The structure $\mathcal{A}$ with $U^{\mathcal{A}} = \mathbb{Z}_3$ and $P^{\mathcal{A}}(x,y) = 1$ if and only if $x + 1 \equiv_3 y$. $\mathcal{A}$ is a model for G, because (1) for any x there exists a $y = R_3(x+1)$ such that $x+1 \equiv_3 y$ and similarly for any y there exists an $x = R_3(y-1)$ such that $x + 1 \equiv_3 y$, and (2) if $x + 1 \equiv_3 y$ then $y + 1 \equiv_3 x + 2 \not\equiv_3 x$.

13.3 Predicate Logic with Equality

a) An interpretation $\mathcal{A}$ is a model for F if and only if $|U^{\mathcal{A}}| = 1$.

If $|U^{\mathcal{A}}| = 1$, then clearly for all elements x, y of the universe, we have $x = y$ and $\mathcal{A}$ is a model for F. On the other hand, if $U^{\mathcal{A}}$ contains at least two different elements, then $\mathcal{A}$ is not a model, because there exists x and y such that $\neg(x = y)$.

b) An interpretation $\mathcal{A}$ is a model for G if and only if $|U^{\mathcal{A}}| > 1$.

If $|U^{\mathcal{A}}| > 1$, then there exist two different elements x, y of the universe and $\mathcal{A}$ is a model for G. On the other hand, if $|U^{\mathcal{A}}| = 1$, then $\mathcal{A}$ is not a model, because for all x, y, we have $x = y$.

c) An example of such formula H is $\exists x \exists y \exists z \big(\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z)\big)$.

If $|U^{\mathcal{A}}| \geq 3$, then there exist three different elements x, y, z of the universe. These elements satisfy $\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z)$.

If $|U^{\mathcal{A}}| < 3$, then, by the pigeonhole principle, at least two among three elements chosen from the universe must be equal. Hence, at least one of $\neg(x = y), \neg(y = z)$ and $\neg(x = z)$ must be false and $\mathcal{A}(H) = 0$.

13.4 Statements About Formulas

a) The statement is true.

Proof. Let $\mathcal{A}$ be any interpretation suitable for both $\forall x\ (F \wedge G)$ and $(\forall x\ F) \wedge G$, such that $\mathcal{A}\left(\forall x\ (F \wedge G)\right) = 1$. According to the semantics of $\forall$, we have $\mathcal{A}_{[x \to u]}(F \wedge G) = 1$ for all $u \in U$. According to the semantics of $\wedge$, we further have (1) $\mathcal{A}_{[x \to u]}(F) = 1$ for all $u \in U$ and (2) $\mathcal{A}_{[x \to u]}(G) = 1$ for all $u \in U$.

The fact (1) implies (3) $\mathcal{A}(\forall x\ F) = 1$, according to the semantics of $\forall$. Furthermore, note that if x appears free in G, then it also appears free in $(\forall x\ F) \wedge G$, and since $\mathcal{A}$ is suitable for $(\forall x\ F) \wedge G$, it must assign a value to x. We now define u^* as follows: if x appears free in G, then u^* is the value assigned to x by $\mathcal{A}$, else u^* is arbitrary. By the definition of u^*, we have $\mathcal{A}_{[x \to u^*]}(G) = \mathcal{A}(G)$, so by (2), we have (4) $\mathcal{A}(G) = 1$.

The facts (3) and (4) imply that $\mathcal{A}\left((\forall x\ F) \wedge G\right) = 1$.

b) The statement is false.

Counterexample. Let $F = P(x)$ and $G = Q(x)$. Let $\mathcal{A}$ be the interpretation with the universe $U^{\mathcal{A}} = \{0, 1\}$, which defines:

- $P^{\mathcal{A}}(0) = 1$ and $P^{\mathcal{A}}(1) = 1$
- $Q^{\mathcal{A}}(0) = 1$ and $Q^{\mathcal{A}}(1) = 0$
- $x^{\mathcal{A}} = 1$

We then have $\mathcal{A}(\exists x\ (P(x) \wedge Q(x))) = 1$, because $\mathcal{A}_{[x \to 0]}(P(x) \wedge Q(x)) = 1$. However, $\mathcal{A}((\exists x\ P(x)) \wedge Q(x)) = 0$, because $\mathcal{A}(Q(x)) = 0$.

13.5 More Statements About Formulas

a) The statement is true. Let $\mathcal{A}$ be any interpretation suitable for $\forall x\ (F \to G)$ and $(\forall x\ F) \to (\forall x\ G)$. Assume $\mathcal{A}(\forall x\ (F \to G)) = 1$. Case distinction:

- $\mathcal{A}(\forall x\ F) = 0$. Then, $\mathcal{A}(\neg(\forall x\ F)) = 1$.
- $\mathcal{A}(\forall x\ F) = 1$. Let $u \in U^{\mathcal{A}}$ be arbitrary. We have $\mathcal{A}_{[x\to u]}(F) = 1$. Moreover,

$$
\begin{aligned}
& \mathcal{A}(\forall x\ (F \to G)) = 1 && \\
\Longrightarrow\ & \mathcal{A}(\forall x\ (\neg F \vee G)) = 1 && \text{(def. } \to\text{)} \\
\Longrightarrow\ & \mathcal{A}_{[x\to u]}(\neg F \vee G) = 1 && \text{(sem. } \forall\text{)} \\
\Longrightarrow\ & \mathcal{A}_{[x\to u]}(\neg F) = 1 \text{ or } \mathcal{A}_{[x\to u]}(G) = 1 && \text{(sem. } \vee\text{)} \\
\Longrightarrow\ & \mathcal{A}_{[x\to u]}(F) = 0 \text{ or } \mathcal{A}_{[x\to u]}(G) = 1 && \text{(sem. } \neg\text{)} \\
\Longrightarrow\ & \mathcal{A}_{[x\to u]}(G) = 1. && (\mathcal{A}_{[x\to u]}(F) = 1)
\end{aligned}
$$

  Since u was arbitrary, we obtain $\mathcal{A}(\forall x\ G) = 1$.

Combining both cases, we obtain $\mathcal{A}((\forall x\ F) \to (\forall x\ G)) = \mathcal{A}(\neg(\forall x\ F) \vee (\forall x\ G)) = 1$ by the semantics of $\vee$. Hence, $\forall x\ (F \to G) \models\ (\forall x\ F) \to (\forall x\ G)$.

b) The statement is false. As a counterexample, consider the formulas $F = P(x)$ and $G = \neg P(x)$, and the interpretation $\mathcal{A}$ with $U^{\mathcal{A}} = \{0, 1\}, P^{\mathcal{A}}(x) = 1 \iff x = 1$.
Observe that $\mathcal{A}(\forall x\ F) = 0$ since $\mathcal{A}_{[x\to 0]}(P(x)) = 0$ and therefore $\mathcal{A}(\neg(\forall x\ F)) = 1$. Thus, the semantics of $\vee$ implies $\mathcal{A}((\forall x\ F) \to (\forall x\ G)) = \mathcal{A}(\neg(\forall x\ F) \vee (\forall x\ G)) = 1$.
Moreover, we have $\mathcal{A}_{[x\to 1]}(\neg F \vee G) = \mathcal{A}_{[x\to 1]}(\neg P(x) \vee \neg P(x)) = \mathcal{A}_{[x\to 1]}(\neg P(x)) = 0$ (as $\mathcal{A}_{[x\to 1]}(P(x)) = 1$). Thus, $\mathcal{A}(\forall x\ (F \to G)) = \mathcal{A}(\forall x\ (\neg F \vee G)) = 0$.
Hence, $(\forall x\ F) \to (\forall x\ G) \not\models \forall x\ (F \to G)$.