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Diskrete Mathematik

Solution 6

6.1 An Equivalence Relation

a) We prove that $\sim$ satisfies all properties of an equivalence relation.

Reflexivity: For any point $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have $(x, y) \sim (x, y)$, because one can choose $\lambda = 1$ in the definition of $\sim$.

Symmetry: Let $x_1, y_1, x_2, y_2 \in \mathbb{R} \setminus \{0\}$ and assume that $(x_1, y_1) \sim (x_2, y_2)$. It follows that $x_1 = \lambda x_2$ and $y_1 = \lambda y_2$ for some $\lambda > 0$. Hence, $x_2 = \frac{1}{\lambda} x_1$ and $y_2 = \frac{1}{\lambda} y_1$, where $\frac{1}{\lambda} > 0$. Therefore, $(x_2, y_2) \sim (x_1, y_1)$.

Transitivity: Let $x_1, y_1, x_2, y_2, x_3, y_3 \in \mathbb{R} \setminus \{0\}$ and assume that $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$. This means that $(x_1, y_1) = (\lambda_1 x_2, \lambda_1 y_2)$ and $(x_2, y_2) = (\lambda_2 x_3, \lambda_2 y_3)$ for some $\lambda_1, \lambda_2 > 0$. It follows that $(x_1, y_1) = (\lambda x_3, \lambda y_3)$, where $\lambda > 0$ is defined as $\lambda_1 \lambda_2$. Hence, $(x_1, y_1) \sim (x_3, y_3)$.

b) An equivalence class $[(x, y)]_\sim$ contains all points on the ray through the origin $(0, 0)$ and the point (x, y) (excluding the origin). Note that no equivalence class can contain the origin $(0, 0)$ ($\sim$ is only defined on $\mathbb{R}^2 \setminus \{(0, 0)\}$).

6.2 Composition of Equivalence Relations

We prove that $\rho \circ \sigma$ satisfies all properties of an equivalence relation.

Reflexivity: For any $a \in A$, we have $a \; \rho \; a$ and $a \; \sigma \; a$ by the reflexivity of ρ and σ. Hence, $a \; (\rho \circ \sigma) \; a$ by the definition of $\circ$.

Transitivity: For any $a, b, c \in A$, we have

$$
\begin{aligned}
&(a,b),(b,c) \in \rho \circ \sigma \\
&\quad \implies (a,x) \in \rho, (x,b) \in \sigma, (b,y) \in \rho \text{ and } (y,c) \in \sigma && \text{(def. of } \circ) \\
&\quad\quad\ \text{for some } x, y \in A \\
&\quad \implies (a,x) \in \rho, (x,y) \in \sigma \circ \rho \text{ and } (y,c) \in \sigma && \text{(def. of } \circ) \\
&\quad\quad\ \text{for some } x, y \in A \\
&\quad \implies (a,x) \in \rho, (x,y) \in \rho \circ \sigma \text{ and } (y,c) \in \sigma && (\rho \circ \sigma = \sigma \circ \rho) \\
&\quad\quad\ \text{for some } x, y \in A \\
&\quad \implies (a,x) \in \rho, (x,z) \in \rho \text{ and } (z,y) \in \sigma, (y,c) \in \sigma && \text{(def. of } \circ) \\
&\quad\quad\ \text{for some } x, y, z \in A \\
&\quad \implies (a,z) \in \rho \text{ and } (z,c) \in \sigma \text{ for some } z \in A && \text{(trans. of } \rho, \sigma) \\
&\quad \implies (a,c) \in \rho \circ \sigma && \text{(def. of } \circ)
\end{aligned}
$$

Symmetry: For any $a, b \in A$, we have

$$
\begin{aligned}
&(a,b) \in \rho \circ \sigma \\
&\quad \iff (a,b) \in \sigma \circ \rho && (\rho \circ \sigma = \sigma \circ \rho) \\
&\quad \iff (a,x) \in \sigma \text{ and } (x,b) \in \rho \text{ for some } x \in A && \text{(def. of } \circ) \\
&\quad \iff (x,a) \in \sigma \text{ and } (b,x) \in \rho \text{ for some } x \in A && \text{(symm. of } \rho, \sigma) \\
&\quad \iff (b,a) \in \rho \circ \sigma && \text{(def. of } \circ)
\end{aligned}
$$

6.3 Lifting an Operation to Equivalence Classes

a) We define the function $\mathsf{sum} : A^2 \to A$ by

$$\mathsf{sum}((a,b),(c,d)) \stackrel{\text{def}}{=} (ad + bc, bd).$$

Observe that $bd \neq 0$ since $b \neq 0$ and $d \neq 0$.

b) f is θ-consistent if and only if

$$(b_1 \; \theta \; b_1' \text{ and } b_2 \; \theta \; b_2') \implies f(b_1, b_2) \; \theta \; f(b_1', b_2')$$

is true for all $b_1, b_2, b_1', b_2' \in B$. Alternatively (and equivalently) we could say that f is θ-consistent if and only if

$$([b_1]_\theta = [b_1']_\theta \text{ and } [b_2]_\theta = [b_2']_\theta) \implies [f(b_1, b_2)]_\theta = [f(b_1', b_2')]_\theta$$

is true for all $b_1, b_2, b_1', b_2' \in B$.

c) Let $(a,b),(a',b'),(c,d),(c',d') \in A$ be arbitrary. We have

$$\begin{aligned}
(a,b) \sim (a',b') &\text{ and } (c,d) \sim (c',d') \\
&\iff ab' = ba' \text{ and } cd' = dc' && \text{(def. } \sim\text{)} \\
&\implies ab' \cdot dd' + cd' \cdot bb' = ba' \cdot dd' + dc' \cdot bb' \\
&\iff ad \cdot b'd' + bc \cdot b'd' = bd \cdot a'd' + bd \cdot b'c' && \text{(comm.)} \\
&\iff (ad+bc) \cdot b'd' = bd \cdot (a'd' + b'c') && \text{(distr.)} \\
&\iff (ad+bc, bd) \sim (a'd'+b'c', b'd') && \text{(def. } \sim\text{)} \\
&\iff \mathsf{sum}((a,b),(c,d)) \sim \mathsf{sum}((a',b'),(c',d')). && \text{(def. sum)}
\end{aligned}$$

Hence, sum is $\sim$-consistent.

6.4 Partial Order Relations

a) **i)** 11 and 12 are incomparable, since $11 \nmid 12$ and $12 \nmid 11$.
ii) 4 and 6 are incomparable, since $4 \nmid 6$ and $6 \nmid 4$.
iii) 5 and 15 are comparable, since $5 \mid 15$.
iv) 42 and 42 are comparable, since $42 \mid 42$.

b) The elements $(a,b) \in A$, such that $(a,b) \leq_{\mathsf{lex}} (2,5)$ are: $(2,1),(2,5)$ and $(1,n)$ for all $n \in \mathbb{N} \setminus \{0\}$.

Justification: Let $(a,b) \in A$. We distinguish the following cases:

Case $a = 1$: Since $1 \mid 2$, we have $(a,b) \leq_{\mathsf{lex}} (2,5)$ for any b.

Case $a = 2$: Since 1 and 5 are the only natural numbers which divide 5, we have $(a,b) \leq_{\mathsf{lex}} (2,5)$ only for $b \in \{1,5\}$.

Case $a > 2$: Since $a \nmid 2$, $(a,b) \leq_{\mathsf{lex}} (2,5)$ cannot hold for any b.

c) $(\{1,3,6,9,12\};\ |\ )$ is not a lattice, since 9 and 12 do not have a common upper bound.

d) $(A; \widehat{\preceq})$ is a poset. To prove this, we show that $\widehat{\preceq}$ is a partial order on A.

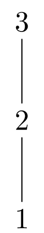
Reflexivity: For any $a \in A$, by the reflexivity of $\preceq$, we have $a \preceq a$, hence, $a \widehat{\preceq} a$.

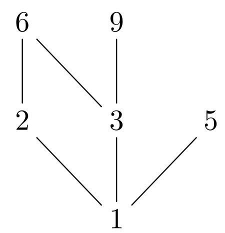
Antisymmetry: Let $a, b \in A$ be such that $a \widehat{\preceq} b$ and $b \widehat{\preceq} a$. This means that $b \preceq a$ and $a \preceq b$ By the antisymmetry of $\preceq$, it follows that $a = b$.

Transitivity: Let $a,b,c \in A$ be such that $a \widehat{\preceq} b$ and $b \widehat{\preceq} c$. This means that $b \preceq a$ and $c \preceq b$. By the transitivity of $\preceq$, we have $c \preceq a$. Hence, $a \widehat{\preceq} c$.

6.5 Hasse Diagrams

a) The Hasse diagrams of the posets $(\{1,2,3\}; \leq)$ and $(\{1,2,3,5,6,9\};\ |\ )$ are as follows:





In both cases, 1 is the least and the only minimal element. In the poset $(\{1,2,3\};\leq)$, the greatest and the only maximal element is 3. In the poset $(\{1,2,3,5,6,9\};\ |\ )$ there is no greatest element. The maximal elements in this poset are $5, 6$ and 9.

6.6 The Lexicographic Order

For posets $(A;\preceq)$ and $(B;\sqsubseteq)$ the lexicographic order $\leq_{\mathsf{lex}}$ on $A \times B$ is defined by

$$(a_1, b_1) \leq_{\mathsf{lex}} (a_2, b_2) \ :\Longleftrightarrow\ a_1 \prec a_2 \vee (a_1 = a_2 \wedge b_1 \sqsubseteq b_2)$$

We show that $\leq_{\mathsf{lex}}$ is a partial order relation.

Reflexivity: Take any $(a_1, b_1) \in A \times B$. Since $\sqsubseteq$ is reflexive, we have $b_1 \sqsubseteq b_1$. Hence, it is true that $(a_1 = a_1 \wedge b_1 \sqsubseteq b_1)$ and, thus, $(a_1, b_1) \leq_{\mathsf{lex}} (a_1, b_1)$.

Antisymmetry: Take any (a_1, b_1) and (a_2, b_2) in $A \times B$ such that $(a_1, b_1) \leq_{\mathsf{lex}} (a_2, b_2)$ and $(a_2, b_2) \leq_{\mathsf{lex}} (a_1, b_1)$. This means that

$$\underbrace{a_1 \prec a_2}_{(1)} \vee \underbrace{(a_1 = a_2 \wedge b_1 \sqsubseteq b_2)}_{(2)} \quad \text{and} \quad \underbrace{a_2 \prec a_1}_{(3)} \vee \underbrace{(a_2 = a_1 \wedge b_2 \sqsubseteq b_1)}_{(4)}.$$

We have to show that $(a_1, b_1) = (a_2, b_2)$. The proof proceeds by case distinction.

(1) **and** (3)**:** We have $a_1 \preceq a_2 \ \wedge\ a_1 \neq a_2$ and $a_2 \preceq a_1 \ \wedge\ a_2 \neq a_1$. But since $\preceq$ is antisymmetric, it follows that $a_1 = a_2$, which is a contradiction with $a_1 \neq a_2$. Therefore, this case cannot occur.

(1) **and** (4)**:** We have $a_1 \preceq a_2 \wedge a_1 \neq a_2$ and $a_2 = a_1 \wedge b_2 \sqsubseteq b_1$, which is a contradiction. Therefore, this case also cannot occur.

(2) **and** (3)**:** We have $a_1 = a_2 \wedge b_1 \sqsubseteq b_2$ and $a_2 \preceq a_1 \wedge a_2 \neq a_1$, which is a contradiction. Therefore, this case cannot occur as well.

(2) **and** (4)**:** We have $a_1 = a_2 \wedge b_1 \sqsubseteq b_2$ and $a_2 = a_1 \wedge b_2 \sqsubseteq b_1$. Since $\sqsubseteq$ is antisymmetric, it follows that $b_1 = b_2$. But we also have $a_1 = a_2$ and, thus, $(a_1, b_1) = (a_2, b_2)$.

Transitivity: Take any $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ in $A \times B$ such that $(a_1, b_1) \leq_{\mathsf{lex}} (a_2, b_2)$ and $(a_2, b_2) \leq_{\mathsf{lex}} (a_3, b_3)$. This means that

$$\underbrace{a_1 \prec a_2}_{(1)} \vee \underbrace{(a_1 = a_2 \wedge b_1 \sqsubseteq b_2)}_{(2)} \quad \text{and} \quad \underbrace{a_2 \prec a_3}_{(3)} \vee \underbrace{(a_2 = a_3 \wedge b_2 \sqsubseteq b_3)}_{(4)}.$$

We have to show that $(a_1, b_1) \leq_{\mathsf{lex}} (a_3, b_3)$. The proof proceeds by case distinction.

(1) **and** (3)**:** We have $a_1 \prec a_2$ and $a_2 \prec a_3$. Since $\preceq$ is transitive we have $a_1 \preceq a_3$. Moreover, if we had $a_1 = a_3$, the antisymmetry of $\preceq$ would imply that $a_1 = a_2$, a contradiction to $a_1 \prec a_2$. Thus, $a_1 \neq a_3$, and therefore $a_1 \prec a_3$. Hence, $(a_1, b_1) \leq_{\mathsf{lex}} (a_3, b_3)$.

(1) **and** (4)**:** We have $a_1 \prec a_2$ and $a_2 = a_3 \wedge b_2 \sqsubseteq b_3$. Hence, $a_1 \prec a_3$ and, therefore, $(a_1, b_1) \leq_{\mathsf{lex}} (a_3, b_3)$.

(2) **and** (3)**:** We have $a_1 = a_2 \wedge b_1 \sqsubseteq b_2$ and $a_2 \prec a_3$. Hence, $a_1 \prec a_3$ and, therefore, $(a_1, b_1) \leq_{\mathsf{lex}} (a_3, b_3)$.

(2) **and** (4)**:** We have $a_1 = a_2 \wedge b_1 \sqsubseteq b_2$ and $a_2 = a_3 \wedge b_2 \sqsubseteq b_3$. It follows that $a_1 = a_3$. Since $\sqsubseteq$ is transitive, we also have $b_1 \sqsubseteq b_3$. Therefore, $(a_1, b_1) \leq_{\mathsf{lex}} (a_3, b_3)$.

6.7 Inverses of Functions

We prove the two implications separately.

($\Longrightarrow$) Let g be a function such that $g \circ f = \mathrm{id}$. We show that f is injective. Assume that $f(a) = f(b)$ for some $a, b \in A$. Then

$$
\begin{aligned}
a &= (g \circ f)(a) && (g \circ f = \mathrm{id}) \\
&= g(f(a)) && (\text{def. } \circ) \\
&= g(f(b)) && (f(a) = f(b)) \\
&= (g \circ f)(b) && (\text{def. } \circ) \\
&= b && (g \circ f = \mathrm{id})
\end{aligned}
$$

($\Longleftarrow$) Assume that f is injective. We construct a function g such that $g \circ f = \mathrm{id}$ as follows. For any $b \in \mathrm{Im}(f)$, by the injectivity of f, there exists a unique a such that $f(a) = b$, and we define $g(b) = a$. For $b \notin \mathrm{Im}(f)$, we define $g(b) = b$. We have $g \circ f = \mathrm{id}$, because for any $a \in A$, $f(a) \in \mathrm{Im}(f)$, so $g(f(a)) = a$.

Note: The choice $g(b) = b$ in case $b \notin \mathrm{Im}(f)$ is irrelevant. For example, we could set $g(b) = a_0$ for some fixed $a_0 \in A$.