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Diskrete Mathematik

Solution 5

5.1 A Property of Any Two Sets

We prove the statement constructively. Let A and B be two sets. We define

$$C \stackrel{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

In order to show that $A = (B \setminus C) \cup (C \setminus B)$ we first show that for any x we have

$$x \in B \setminus C \iff x \in B \wedge x \in A \tag{1}$$

and

$$x \in C \setminus B \iff x \notin B \wedge x \in A \tag{2}$$

Combining both claims we obtain

$$\begin{aligned}
& x \in (B \setminus C) \cup (C \setminus B) \\
& \quad \iff x \in (B \setminus C) \vee x \in (C \setminus B) && \text{(def. } \cup\text{)} \\
& \quad \iff (x \in B \wedge x \in A) \vee (x \notin B \wedge x \in A) && \text{((1) and (2))} \\
& \quad \iff (x \in B \vee x \notin B) \wedge x \in A && \text{(distributivity)} \\
& \quad \iff x \in A. && (F \vee \neg F \equiv \top, \top \wedge G \equiv G)
\end{aligned}$$

It thus only remains to prove (1) and (2). We first prove (1).

$$\begin{aligned}
& x \in B \setminus C \\
& \quad \iff x \in B \wedge x \notin C \\
& \quad \iff x \in B \wedge \neg(x \in (A \setminus B) \cup (B \setminus A)) && \text{(def. } C\text{)} \\
& \quad \iff x \in B \wedge \neg(x \in (A \setminus B) \vee x \in (B \setminus A)) && \text{(def. } \cup\text{)} \\
& \quad \iff x \in B \wedge \neg((x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)) && \text{(def. } \setminus\text{)} \\
& \quad \iff x \in B \wedge \neg(x \in A \wedge x \notin B) \wedge \neg(x \in B \wedge x \notin A) && \text{(de Morgan)} \\
& \quad \iff x \in B \wedge (x \notin A \vee x \in B) \wedge (x \notin B \vee x \in A) && \text{(de Morgan)} \\
& \quad \iff x \in B \wedge (x \notin B \vee x \in A) && \text{(absorption)} \\
& \quad \iff (x \in B \wedge x \notin B) \vee (x \in B \wedge x \in A) && \text{(distributivity)} \\
& \quad \iff x \in B \wedge x \in A && (F \wedge \neg F \equiv \bot, \bot \vee G \equiv G)
\end{aligned}$$

Now, we prove (2).

$$
\begin{aligned}
& x \in C \setminus B \\
& \iff x \in C \land x \notin B \\
& \iff x \in (A \setminus B) \cup (B \setminus A) \land x \notin B && \text{(def. } C\text{)} \\
& \iff (x \in A \setminus B \lor x \in B \setminus A) \land x \notin B && \text{(def. } \cup\text{)} \\
& \iff ((x \in A \land x \notin B) \lor (x \in B \land x \notin A)) \land x \notin B && \text{(def. } \setminus\text{)} \\
& \iff ((x \in A \land x \notin B) \land x \notin B) \lor ((x \in B \land x \notin A) \land x \notin B) && \text{(distributivity)} \\
& \iff (x \in A \land x \notin B \land x \notin B) \lor ((x \in B \land x \notin B) \land x \notin A) && \text{(assoc., comm.)} \\
& \iff (x \in A \land x \notin B \land x \notin B) \lor \bot && (F \land \neg F \equiv \bot, \bot \land G \equiv \bot) \\
& \iff x \in A \land x \notin B \land x \notin B && (F \lor \bot \equiv F) \\
& \iff x \notin B \land x \in A && \text{(idempotence, comm.)}
\end{aligned}
$$

This concludes the proof.

Alternative proof. We show a less formal proof of $A = (B \setminus C) \cup (C \setminus B)$.
We first prove $A \subseteq (B \setminus C) \cup (C \setminus B)$. Assume $x \in A$. We do a case distinction on $x \in B$.

- $x \in B$. Then, we have $x \notin (A \setminus B)$ and $x \notin (B \setminus A)$, so $x \notin C$. Thus, $x \in (B \setminus C)$, and therefore $x \in (B \setminus C) \cup (C \setminus B)$.
- $x \notin B$. Then, we have $x \in (A \setminus B)$, so $x \in (A \setminus B) \cup (B \setminus A) = C$. Thus, $x \in (C \setminus B)$, and therefore $x \in (B \setminus C) \cup (C \setminus B)$.

It remains to prove that $(B \setminus C) \cup (C \setminus B) \subseteq A$. Assume $x \in (B \setminus C) \cup (C \setminus B)$. We have $x \in (B \setminus C)$ or $x \in (C \setminus B)$. Case distinction.

- $x \in (B \setminus C)$. Then, $x \in B$ and $x \notin C$. By definition of C we have $x \notin (B \setminus A)$. Moreover, by definition of set difference (and De Morgan), $x \notin (B \setminus A)$ implies $x \notin B$ or $x \in A$. Combined with $x \in B$, this implies $x \in A$.
- $x \in (C \setminus B)$. Then, $x \in C$ and $x \notin B$. Since we have by definition of C either $x \in (A \setminus B)$ or $x \in (B \setminus A)$. Since $x \notin B$ implies $x \notin (B \setminus A)$ we must have $x \in (A \setminus B)$. Hence, $x \in A$.

Combining both directions, we have proved that $A = (B \setminus C) \cup (C \setminus B)$.

5.2 Relating Two Power Sets

a) For any C, we have

$$
\begin{aligned}
& C \in \mathcal{P}(A \cap B) \\
& \quad \Longleftrightarrow \quad C \subseteq A \cap B && \text{(definition of } \mathcal{P}) \\
& \quad \Longleftrightarrow \quad \forall c \, \big(c \in C \to c \in A \cap B\big) && \text{(definition of } \subseteq) \\
& \quad \Longleftrightarrow \quad \forall c \, \big(c \in C \to (c \in A \land c \in B)\big) && \text{(definition of } \cap) \\
& \quad \Longleftrightarrow \quad \forall c \, \big((c \in C \to c \in A) \land (c \in C \to c \in B)\big) && (*) \\
& \quad \Longleftrightarrow \quad \forall c \, (c \in C \to c \in A) \land \forall c \, (c \in C \to c \in B) && (**) \\
& \quad \Longleftrightarrow \quad C \subseteq A \land C \subseteq B && \text{(definition of } \subseteq) \\
& \quad \Longleftrightarrow \quad C \in \mathcal{P}(A) \land C \in \mathcal{P}(B) && \text{(definition of } \mathcal{P}) \\
& \quad \Longleftrightarrow \quad C \in \mathcal{P}(A) \cap \mathcal{P}(B) && \text{(definition of } \cap)
\end{aligned}
$$

$(*)$ We use the fact that for any formulas A_1, A_2 and A_3, we have $A_1 \to (A_2 \land A_3) \equiv \neg A_1 \lor (A_2 \land A_3) \equiv (\neg A_1 \lor A_2) \land (\neg A_1 \lor A_3) \equiv (A_1 \to A_2) \land (A_1 \to A_3)$. (This follows from Lemma 2.1.)

$(**)$ We use the fact that $\forall x P(x) \land \forall x Q(x) \equiv \forall x (P(x) \land Q(x))$ for any predicates P and Q (see Chapter 2.4.8 of the lecture notes).

b) To prove that the statement is false, we show a counterexample. Let $A = \{1\}$ and $B = \{2\}$. We have $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, \{1\}\} \cup \{\varnothing, \{2\}\} = \{\varnothing, \{1\}, \{2\}\}$. On the other hand, $\mathcal{P}(A \cup B) = \mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

c) We will prove the implication in both directions separately.

$A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$: Let B be any set and let A be any subset of B. What we have to show is that each element of $\mathcal{P}(A)$ is also an element of $\mathcal{P}(B)$. Let S be any element of $\mathcal{P}(A)$. Then, by Definition 3.5, $S \subseteq A$. By the assumption that $A \subseteq B$ and by the transitivity of $\subseteq$, it follows that $S \subseteq B$. This means that S is an element of $\mathcal{P}(B)$.

$\mathcal{P}(A) \subseteq \mathcal{P}(B) \implies A \subseteq B$: Let A, B be any sets and assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \in \mathcal{P}(A)$ (which holds for any set A) and, by assumption, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have that $A \in \mathcal{P}(B)$. By Definition 3.5, this means that $A \subseteq B$.

5.3 Family Relations

a) The relations can be expressed in the following way:

i) $\mathit{iggf} = \mathit{if} \circ \mathit{ip} \circ \mathit{ip}$

ii) $\mathit{ihs} = (\mathit{ic} \circ \mathit{ip}) \setminus (\mathit{ic} \circ \mathit{im} \cap \mathit{ic} \circ \mathit{if})$

iii) $\mathit{ico} = (\mathit{ic} \circ \mathit{ic} \circ \mathit{ip} \circ \mathit{ip}) \setminus (\mathit{ic} \circ \mathit{ip})$

b) These relations are neither the same, nor is one a subset of the other. To see this, consider six different people a, b, c, d, e and f, where c and d are the mother and father of a, while e and f are the mother and father of b. Also, c has no common parent or child with e or f and f has no common parent or child with c or d.

For such six people, a *ic* ∘ *ic* ∘ *ip* ∘ *ip* b if and only if d and e have a common parent, while a *ic* ∘ *ip* ∘ *ic* ∘ *ip* b if and only if d and e have a common child.

While it is theoretically possible that two people share both a parent and a child, in general neither of these implies the other. One can easily argue that a counterexample actually exists.

5.4 Computing Representations of Relations

a) We have $\rho^3 = \{(1,1),(1,3),(2,2),(4,4)\}$ and

$$M^{\rho^*} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

5.5 Operations on Relations

	Relation	reflexive	symmetric	transitive
a)	$< \circ \mid$	✗	✗	✓
b)	$\mid \cup \equiv_2$	✓	✗	✗
c)	$\mid \cup \mid^{-1}$	✓	✓	✗

a) Two numbers (a, b) are in the relation whenever there exists an x such that $a < x$ and $x \mid b$. This relation is not reflexive, since $(1,1) \notin < \circ \mid$. Moreover, it is not symmetric, because $(1,2) \in < \circ \mid$, but $(2,1) \notin < \circ \mid$. This relation is transitive. For any (a,b,c), assume that there exist some x and y, such that $a < x,\ x \mid b,\ b < y$ and $y \mid c$. From $x \mid b$ it follows that $x \leq b$, hence, $a < x \leq b < y$. Therefore, $a < y$ and $y \mid c$.

b) Two numbers (a, b) are in the relation whenever $a \mid b$ or $a \equiv_2 b$. This relation is reflexive, since for any a, we have $a \equiv_2 a$ (alternatively, one could use the fact that $a \mid a$). It is, however, not symmetric, because $(1,2) \in\ \mid \cup \equiv_2$, but $(2,1) \notin\ \mid \cup \equiv_2$. It is also not transitive, since $(3,1) \in\ \mid \cup \equiv_2$ and $(1,2) \in\ \mid \cup \equiv_2$, but $(3,2) \notin\ \mid \cup \equiv_2$.

c) Two numbers (a, b) are in the relation whenever $a \mid b$ or $b \mid a$. This relation is reflexive, since for any a, we have $a \mid a$. It is also symmetric, because for any (a,b), we trivially have $a \mid b$ or $b \mid a$ if and only if $b \mid a$ or $a \mid b$. The relation is, however, not transitive, since $(3,1) \in \mid \cup \mid^{-1}$ and $(1,2) \in \mid \cup \mid^{-1}$ but $(3,2) \notin \mid \cup \mid^{-1}$.

5.6 A False Proof

a) For an arbitrary $x \in A$, there does not always exist a $y \in A$ such that $x \ \rho \ y$.

b) Consider the following counterexample: $A = \{1,2\}$ and $\rho = \{(1,1)\}$. The relation ρ is symmetric and transitive. However, it is not reflexive, since $2 \ \rho \ 2$ does not hold.

5.7 Properties of Relations

a) The statement is false. A counterexample is the relation $\sigma = \{(0,1),(1,0)\}$ on the set $A = \{0,1\}$. Obviously, σ is not reflexive. Further, we have $\sigma^2 = \{(0,0),(1,1)\}$, which is reflexive. This disproves the statement.

b) The statement is true. We present two different (direct) proofs, each using one of the two equivalent definitions of an antisymmetric relation.

Proof 1 Let A be any set and let σ and ρ be any antisymmetric relations on A. We show that $\sigma \cap \rho$ is antisymmetric, that is that for any $(a,b) \in A \times A$, we have

$$a\ (\sigma \cap \rho)\ b \ \ \wedge \ \ b\ (\sigma \cap \rho)\ a \ \ \Longrightarrow \ \ a = b.$$

To this end, consider any pair $(a,b) \in A \times A$ such that $a\ (\sigma \cap \rho)\ b$ and $b\ (\sigma \cap \rho)\ a$. From $a\ (\sigma \cap \rho)\ b$ it follows that $a\ \sigma\ b$. Moreover, from $b\ (\sigma \cap \rho)\ a$ it follows that $b\ \sigma\ a$.

From those two facts and the antisymmetry of σ we can conclude that $a = b$.

Proof 2 Let A be any set and let σ and ρ be any antisymmetric relations on A. We show that $\sigma \cap \rho$ is antisymmetric, that is that $(\sigma \cap \rho) \cap (\sigma \cap \rho)^{-1} \subseteq \mathsf{id}$.

$$\begin{aligned}
(\sigma \cap \rho) \cap (\sigma \cap \rho)^{-1} &= \sigma \cap \rho \cap (\sigma \cap \rho)^{-1} \\
&\subseteq \rho \cap (\sigma \cap \rho)^{-1} && (A \cap B \subseteq B) \\
&\subseteq \rho \cap \rho^{-1} && (\alpha \subseteq \beta \implies \alpha^{-1} \subseteq \beta^{-1}) \\
&\subseteq \mathsf{id}, && (\text{antisymmetry of } \rho)
\end{aligned}$$

where the statement $\alpha \subseteq \beta \implies \alpha^{-1} \subseteq \beta^{-1}$ follows directly from the definition of the inverse of a relation. Therefore, $\sigma \cap \rho$ is antisymmetric.