ETH Zürich, D-INFK
HS 2020, 7. October 2020

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Diskrete Mathematik

Solution 3

3.1 Expressing Relationship of Humans in Predicate Logic

a) $\mathtt{m}(x) \land \exists u\, \exists v\ \ (\mathtt{par}(x,u) \land \mathtt{par}(u,v) \land \mathtt{par}(v,y))$

b) $\exists u\, \exists v\ \ (\mathtt{par}(u,x) \land \mathtt{par}(u,y) \land \mathtt{par}(v,x) \land \neg\mathtt{par}(v,y))$

3.2 Quantifiers and Predicates

a) **i)** $\forall m\, \forall n\, \big(0 < m \cdot n \to (0 < m \lor 0 < n)\big)$

This statement is false. For example, $(-2) \cdot (-2) = 4$.

ii) $\forall m\, \big(0 < m \to \exists n\big(0 < n \land m < n \land (\exists k\, n = 3 \cdot k)\big)\big)$

This statement is true. For any n, one of the numbers $n+1$, $n+2$, $n+3$ must be divisible by 3.

In the formula above, we assumed that 0 is not a natural number. An (equally good) solution for the case when 0 is a natural number would be
$\forall m\, \big(-1 < m \to \exists n\big(-1 < n \land m < n \land (\exists k\, n = 3 \cdot k)\big)\big)$

It is also allowed to drop the condition $0 < n$ (respectively, $-1 < n$), since it is implied by $m < n$.

iii) $\forall n\, \big(((\exists k\, n = 2 \cdot k) \land 2 < n) \to \exists p\, \exists q\, (\mathtt{prime}(p) \land \mathtt{prime}(q) \land n = p + q)\big)$

This statement is known as the (strong) Goldbach conjecture. It is not known whether it is true.

b) There are many equally good ways to describe given formulas using words. We only give examples:

i) "For every integer x, there exists an integer y, such that xy is equal to 1."

An alternative solution would be "Each integer has a multiplicative inverse."

This statement is false. For example, there is no integer that will give 1 when multiplied by 5.

ii) "There exists an integer x, such that for all integers y, the product xy is not equal to 1, and such that there exists an integer greater than 0."

This statement is true. For $x = 0$, we have that for any integer y, the product xy is not equal to 1, and that there exists a positive integer, namely 42.

Be careful, the following interpretation is *not* correct (Why?): "There exists an integer x, such that for all integers y, the product xy is not equal to 1 and y is positive."

3.3 Finding an Interpretation for a Formula

a) $U = \mathbb{Z}$ and $P(x, y) = 1 \iff x < y$.

b) $U = \{0, \ldots, n-1\}$ and $P(x, y) = 1 \iff (x < n-1 \land y = x+1) \lor (x = n-1 \land y = 0)$.

3.4 Order of Quantifiers

a) Assume that the formula $\exists y\, \forall x\, P(x, y)$ is true. By the definition of $\exists$, there exists at least one y such that $\forall x\, P(x, y)$ is true. Let y^* be such a y. By the definition of $\forall$, we have that $P(x, y^*)$ is true for all x.
Therefore, we have that for all x there exists a y, namely y^*, for which $P(x, y)$ is true. This means exactly that $\forall x\, \exists y\, P(x, y)$ is true.

b) Consider the following counterexample: the universe is the set $\mathbb{Z}$ of all integers and P is the predicate `less`. In this interpretation, it is true that $\forall x\, \exists y \quad x < y$, but the statement $\exists y\, \forall x \quad x < y$ is false.

3.5 Winning Strategy

a) The numbers announced by Alice cannot depend on Bob's choice for b_1 and b_2. Therefore, the statement can be described by the following formula:

$$\exists a_1 \exists a_2 \forall b_1 \forall b_2 \ \big(a_1 + (a_2 + b_1)^{|b_2|+1} = 1\big).$$

The above statement is false, because for each tuple (a_1, a_2), there exists a tuple $(b_1, b_2) := (2 - a_2 - a_1, 0)$ such that

$$a_1 + (a_2 + b_1)^{|b_2|+1} = a_1 + (a_2 + 2 - a_2 - a_1) = 2.$$

Therefore, Alice does not have a winning strategy.

b) In this case, Alice's choice for a_2 can depend on b_1. Therefore, the statement can be described by the following formula:

$$\exists a_1 \forall b_1 \exists a_2 \forall b_2 \ \big(a_1 + (a_2 + b_1)^{|b_2|+1} = 1\big).$$

This statement is true. A possible winning strategy for Alice is to choose $a_1 = 1$ and $a_2 = -b_1$. For such choice, we have

$$a_1 + (a_2 + b_1)^{|b_2|+1} = 1 + 0^{|b_2|+1} = 1.$$