ETH Zürich, D-INFK
HS 2020, 30. September 2020

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Diskrete Mathematik
Solution 2

2.1 Interpreting Propositional Formulas in Natural Language

a) The formulas can be stated in the English language in the following way:

i) F_1: "The monkey has a banana and does not sit on the palm tree."

ii) F_2: "The monkey sits on the palm tree and has a banana, or it does not sit on the palm tree and it does not have a banana."
Equivalently, we could say "The monkey sits on the palm tree if and only if it has a banana."

b) The sentences can be written formally in the following way:

i) $F_3 = \neg A \wedge \neg B$

ii) $F_4 = (\neg A \wedge B) \vee (A \wedge \neg B)$

c) **i)** $\neg F_3$: The monkey sits on the palm tree or it has a banana.

$$\neg F_3 \;\equiv\; \neg(\neg A \wedge \neg B) \;\equiv A \;\vee B$$

ii) $\neg F_4$: The monkey sits on the palm tree if and only if it has a banana.

$$\neg F_4 \;\equiv\; (A \wedge B) \vee (\neg A \wedge \neg B) \;\equiv\; F_2$$

2.2 Logical Equivalence via Function Tables

a)

A	B	C	$B \to C$	$\neg(A \to C) \wedge \neg(A \vee B)$	$(B \to C) \to \big(\neg(A \to C) \wedge \neg(A \vee B)\big)$
0	0	0	1	0	0
0	0	1	1	0	0
0	1	0	0	0	1
0	1	1	1	0	0
1	0	0	1	0	0
1	0	1	1	0	0
1	1	0	0	0	1
1	1	1	1	0	0

b) With the above function table, it becomes clear that the formula in a) is true if and only if $B \wedge \neg C$ is true. Therefore, the simple equivalent formula is $B \wedge \neg C$.

2.3 Proving Logical Equivalence using Equivalence Transformations

We have:

$$
\begin{aligned}
& (C \wedge A) \vee ((B \rightarrow A) \wedge \neg C) && \\
\equiv\ & (C \wedge A) \vee \big(\neg C \wedge (B \rightarrow A)\big) && \text{(commutativity of } \wedge \text{)} \\
\equiv\ & (C \wedge A) \vee \big(\neg C \wedge (\neg B \vee A)\big) && \text{(definition of } \rightarrow \text{)} \\
\equiv\ & (C \wedge A) \vee \big((\neg C \wedge \neg B) \vee (\neg C \wedge A)\big) && \text{(distributive law } (5)\text{)} \\
\equiv\ & (C \wedge A) \vee \big((\neg C \wedge A) \vee (\neg C \wedge \neg B)\big) && \text{(commutativity of } \vee \text{)} \\
\equiv\ & \big((C \wedge A) \vee (\neg C \wedge A)\big) \vee (\neg C \wedge \neg B)) && \text{(associativity of } \vee \text{)} \\
\equiv\ & \big((A \wedge C) \vee (\neg C \wedge A)\big) \vee (\neg C \wedge \neg B) && \text{(commutativity of } \wedge \text{)} \\
\equiv\ & \big((A \wedge C) \vee (A \wedge \neg C)\big) \vee (\neg C \wedge \neg B) && \text{(commutativity of } \wedge \text{)} \\
\equiv\ & \big(A \wedge (C \vee \neg C)\big) \vee (\neg C \wedge \neg B) && \text{(distributive law } (5)\text{)} \\
\equiv\ & (A \wedge \top) \vee (\neg C \wedge \neg B) && (F \vee \neg F \equiv \top) \\
\equiv\ & A \vee (\neg C \wedge \neg B) && (F \wedge \top \equiv F) \\
\equiv\ & A \vee (\neg B \wedge \neg C) && \text{(commutativity of } \wedge \text{)} \\
\equiv\ & A \vee \neg(B \vee C) && \text{(de Morgan, } \neg(F \vee G) \equiv \neg F \wedge \neg G\text{)}
\end{aligned}
$$

2.4 Logical Consequence

a) We first construct the function table for the formula $A \wedge (A \rightarrow B)$.

A	B	$A \wedge (A \rightarrow B)$
0	0	0
0	1	0
1	0	0
1	1	1

The above table shows that the truth value of $A \wedge (A \rightarrow B)$ is 1 only for the truth assignment in the last row. Clearly, B is also true for that assignment. Thus, B is the logical consequence of $A \wedge (A \rightarrow B)$ and the statement holds.

b) The statement is false. There exists a truth assignment, namely one in which A is false and B is true, for which $A \rightarrow B$ is true, but $\neg A \rightarrow \neg B$ is false.

Thus, $\neg A \rightarrow \neg B$ is not a logical consequence of $A \rightarrow B$.

c) We construct the function table for both formulas: $(A \to B) \wedge (B \to C)$ and $A \to C$.

A	B	C	$A \to B$	$B \to C$	$(A \to B) \wedge (B \to C)$	$A \to C$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	0	1	0	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Analogously to Subtask a), we can show that the statement holds.

2.5 Satisfiability and Tautologies

a) This formula is satisfiable, since it is true for the assignment $A = 0, B = 1$. It is, however, not a tautology, since it is false for the assignment $A = 0, B = 0$.

b) This formula is unsatisfiable (hence, it is not a tautology). In order to prove this, let $F = \big((A \to B) \wedge (B \to C)\big) \wedge \neg(A \to C)$. We notice that

$$\begin{aligned} \neg F &\equiv \neg\big((A \to B) \wedge (B \to C)\big) \vee (A \to C) && \text{(de Morgan's rules)} \\ &\equiv (A \to B) \wedge (B \to C) \;\to\; (A \to C) && \text{(def. } \to\text{)} \end{aligned}$$

From Task 2.4 c), we know that $(A \to B) \wedge (B \to C) \models (A \to C)$ is true. From this fact, together with Lemma 2.3, it follows that $\neg F$ is a tautology. Hence, by Lemma 2.2, F is unsatisfiable.

2.6 Knights and Knaves

Let A be the proposition "The left road leads to the village." and let B be the proposition "The islander is a knight.". We want to ask the islander about the truth value of a formula F in A and B in order to determine whether A is true.

In order to be guaranteed to learn whether A is true or not, we have to receive a fixed answer (say, "Yes") from the islander in case A is true, and the opposite (say, "No") in case A is false. This has to hold *independently* of whether the islander is a knight or a knave (since we have no information about that).

If the islander is a knight (B is true) the answer will be the truth value of F (since knights always tell the truth). However, if the islander is a knave (B is false) the answer will be the truth value of $\neg F$ (since knaves always lie).

Hence, we derive the following partial function table:

A	B	F	$\neg F$
0	0		0
0	1	0	
1	0		1
1	1	1	

This partial function table can be completed (uniquely) to the following function table:

A	B	F	$\neg F$
0	0	1	0
0	1	0	1
1	0	0	1
1	1	1	0

From the function table we obtain a possible formula $F = (\neg A \wedge \neg B) \vee (A \wedge B)$. Formulated as a question: "Does the left road lead to the jungle and you are a knave, or is it the case that the left road leads to the village and you are a knight?".