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Diskrete Mathematik

Solution 1

1.1 The Punctured Chessboard

a) One only needs to consider 10 cases (since all other cases are symmetric). The cases are marked below.

1	2	3	4			
	5	6	7			
		8	9			
			10			

b) The proof that $P(7) = 1$, including all cases, can be found on the following website: `http://www.cut-the-knot.org/Curriculum/Games/TrominoPuzzleN.shtml`.

1.2 A False Proof

More precisely, the claim consists of two parts: "There exists the largest natural number n" and "$n = 1$". Denote the first statement by S and the second by T. The statement to prove is S *and* T, but the proof only shows $S \implies T$, which is true, because S is false. The proof is correct, but proves the wrong statement.

Note that if a statement S is false, then the statement $S \implies T$ is true for any T. In other words, it is possible to prove any statement T by starting with a false assumption.

1.3 Hilbert's Hotel

Generally, there are many valid ways to accommodate the guests. Here we only give one of the many possible solutions.

a) Hilbert asks each guest in room n to move to the room with the number $n + 1$. This way, the room number 1 becomes free and can accommodate Roger Federer.

b) This time, Hilbert asks each guest in room n to move to the room with number $2n$. This way, all rooms with odd numbers become free. A newly arrived guest with number m can now stay in the room number $2m - 1$.

c) Once again, Hilbert asks each guest in room n to move to the room number $2n$. Then, a guest coming in the bus i, with the number m in that bus, takes the room p_{i+1}^m, where p_i denotes the i-th prime (that is, $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$). [1]

Below we argue, a bit more formally, why the above method is correct. The argumentation implicitly uses some proof techniques, such as proof by contradiction and case distinction, as well as some facts from number theory, that will all be explained later during the lecture.

To see why the solution works, we must make sure that the following two statements are true: (1) no new guest is assigned to a room now occupied by an old guest and (2) no two new guests are assigned to the same room. The first statement follows from the fact that 2 is the only even prime (thus, any power of a prime number greater than 2 cannot be even). To justify the second statement, assume that there are two guests: one having the number m in the bus i and the other having the number n in the bus j (where $p_{i+1} \neq p_{j+1}$ or $m \neq n$), who were assigned the same room. That is, assume we have $p_{i+1}^m = p_{j+1}^n$. If it is the case that $p_{i+1} = p_{j+1}$, then it must hold that $m \neq n$ and, thus, $p_{i+1}^m \neq p_{j+1}^n$, which is a contradiction. If $p_{i+1} \neq p_{j+1}$, then we know that a prime power is divisible only by powers of the same prime which is again in contradiction with $p_{i+1}^m = p_{j+1}^n$.

Note that if Hilbert uses the method described above, some rooms are left unoccupied. There exists a more "dense" solution, where all rooms are used. Chapter 3.6 will introduce some techniques that can be used to design such a solution.

[1] A prime is a natural number, which has only two divisors: 1 and itself. Later in the lecture, we will see that there is an infinite number of primes.