Cryptographic Protocols

Spring 2021

MPC Part 3

MPC Active – Information-Theoretic Security

Model

- Active adversary, computationally *unbounded*
- $t < n/3$

Goal

- IT-secure homomorphic commitment scheme
- perfect hiding
- perfect binding

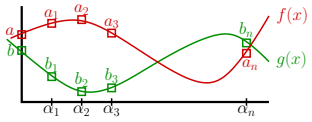
Protocols

- COMMIT
- OPEN
- CTP
- CMP

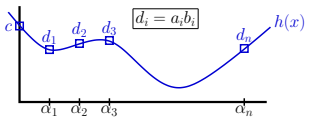
Generic Commitment Multiplication Proof (for $t < n/3$)

0. **Starting point**: D is committed to a, b, c by $\boxed{a}$, $\boxed{b}$, and $\boxed{c}$.

1. **CSP of a, b with degree t**
   $\Rightarrow f(x), g(x)$



2. **CSP of c with degree $2t$**
   use $h(x) = f(x)g(x)$



3. **Checks**
   $\forall P_i$: $d_i \stackrel{?}{=} a_i b_i$, broadcast accusation bit.
   On accusation: Open $\boxed{a_i}$, $\boxed{b_i}$, $\boxed{d_i}$, check $a_i b_i \stackrel{?}{=} d_i$.

Commit State

Open Protocol

1. **Open**
   D broadcasts $g(x)$.
2. **Check consistency**
   P_i accuses dealer if $g(\alpha_i) \neq s_i$.
3. **Compute secret**
   If $\leq t$ accusations: $s = g(0)$.
   If $> t$ accusations: disqualify dealer.

Proof:

Facts about two-dimensional polynomials (1/3)

Given: $f(x, y) = f_{00} + f_{10}x + f_{01}y + f_{11}xy + ... + f_{tt}x^t y^t \in \mathbb{F}[x, y]$
Fact 1: $f_{y_0}(x) := f(x, y_0)$ is a one-dimensional polynomial of degree t.
Proof:

$$\begin{aligned} f(x, y_0) = \quad & (f_{00} + f_{01}y_0 + \ldots + f_{0t}y_0^t) \\ + \ & (f_{10} + f_{11}y_0 + \ldots + f_{1t}y_0^t)x \\ + \ & \ldots \\ + \ & (f_{t0} + f_{t1}y_0 + \ldots + f_{tt}y_0^t)x^t \end{aligned}$$

Facts about two-dimensional polynomials (2/3)

Given: $f(x,y) = f_{00} + f_{10}x + f_{01}y + f_{11}xy + ... + f_{tt}x^t y^t \in \mathbb{F}[x,y]$
Fact 2: Let $X = \{x_1, \ldots, x_{t+1}\}$ and $Y = \{y_1, \ldots, y_{t+1}\}$. Then $f(x,y)$ is uniquely defined by $W := \{(x_i, y_j, z_{ij}) \mid (x_i, y_j) \in X \times Y\}$.
Proof (existence): $\exists \geq 1$ such $f(x,y)$: Lagrange-Interpolation

Find $\lambda_{ij}(x,y)$ with $\begin{cases} \lambda_{ij}(x_i, y_j) & = & 1 \\ \lambda_{ij}(x_{i'}, y_{j'}) & = & 0 \text{ for } (i',j') \neq (i,j) \end{cases}$

$$\Rightarrow \quad \lambda_{ij}(x,y) := \prod_{\substack{i'=1 \\ i' \neq i}}^{t+1} \frac{x - x_{i'}}{x_i - x_{i'}} \prod_{\substack{j'=1 \\ j' \neq j}}^{t+1} \frac{y - y_{j'}}{y_j - y_{j'}}$$

and define

$$f(x,y) := \sum_{i,j=1}^{t+1} \lambda_{ij}(x,y) z_{ij}.$$

Facts about two-dimensional polynomials (3/3)

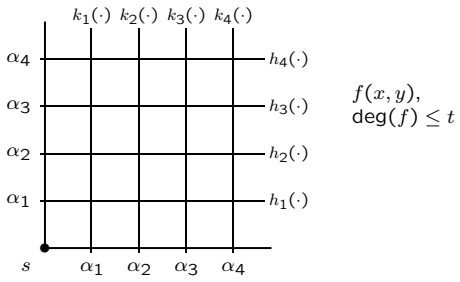
Given: $f(x,y) = f_{00} + f_{10}x + f_{01}y + f_{11}xy + ... + f_{tt}x^t y^t \in \mathbb{F}[x,y]$
Fact 2: Let $X = \{x_1, \ldots, x_{t+1}\}$ and $Y = \{y_1, \ldots, y_{t+1}\}$. Then $f(x,y)$ is uniquely defined by $W := \{(x_i, y_j, z_{ij}) \mid (x_i, y_j) \in X \times Y\}$.
Proof (uniqueness): $\exists \leq 1$ such $f(x,y)$

1. Let $f_1(x,y)$ and $f_2(x,y)$ degree-t-polynomials through W.
2. $f_\Delta(x,y) := f_1(x,y) - f_2(x,y)$ is a degree-t-polynomial.
3. $\forall (x_i, y_j) \in X \times Y : f_\Delta(x_i, y_j) = 0$.
4. $\forall y_j \in Y : f_{y_j}(x) := f_\Delta(x, y_j)$ is a polynomial of degree t (Fact 1).
5. $\forall y_j \in Y : f_{y_j}(x_1) = f_{y_j}(x_2) = \ldots = f_{y_j}(x_{t+1}) = 0$; thus $f_{y_j} \equiv 0$.
6. Thus: $\forall (x, y_j) \in \mathbb{F} \times Y : f_\Delta(x, y_j) = 0$.
7. $\forall x \in \mathbb{F} : f_x(y) := f_\Delta(x,y)$ is a polynomial of degree t (Fact 1).
8. $\forall x \in \mathbb{F} : f_x(y_1) = f_x(y_2) = \ldots = f_x(y_{t+1}) = 0$; thus $f_x \equiv 0$.
9. Thus: $\forall (x,y) \in \mathbb{F} \times \mathbb{F} : f_\Delta(x,y) = 0$; thus $f_\Delta \equiv 0$.

Commit Protocol

1. **Distribution**
   D selects random polynomial
   $$f(x,y) = \sum_{i=0}^{t} \sum_{j=0}^{t} f_{ij}\, x^i y^j, \text{ with } f_{0,0} = s,$$
   and sends $h_i(x) = f(x, \alpha_i)$, $k_i(y) = f(\alpha_i, y)$ to P_i.
2. **Consistency checks**
   $\forall P_i, P_j$: P_i sends $k_i(\alpha_j)$ to P_j, P_j complains if $k_i(\alpha_j) \neq h_j(\alpha_i)$.
   D broadcasts $f(\alpha_i, \alpha_j)$.
3. **Accusation**
   $\forall P_i$: if P_i has received contradicting values from D: accuse D.
   D broadcasts $h_i(x)$ and $k_i(y)$.
   Repeat until no further accusation.
4. **Compute share**
   If $> t$ accusing players: disqualify dealer.
   If $\leq t$ accusing players: $s_i = k_i(0)$.

Commit Protocol



MPC Active, no Setup – Impossibility

MPC Protocols: Summary

Passive Protocol (IT-secure)
- In/Out: Shamir sharing / Lagrange interpolation
- Add: linearity of Shamir sharing
- Mult: local multiplication, Lagrange on sharings

Active Protocol (crypto. secure)
- In/Out: Shamir sharing with commitments, CTP, Lagrange interpolation
- Add: linearity of sharing, homomorph commitments
- Mult: passive, plus CMP that $B^a \sim C$

Active Protocol (IT-secure)
- Commit: 2-dimensional poly, checks, commit-shares on degree-t poly
- In/Out: Shamir sharing with i.t. commit, CTP, Lagrange interpolation
- Add: linearity of sharing, homomorph i.t. commitments
- Mult: passive, plus generic (i.t.) CMP