Cryptographic Protocols

Spring 2021

Part 4

Proofs of Knowledge

Let $Q(\cdot,\cdot)$ be a binary predicate and let a string z be given. Consider the problem of proving knowledge of a secret x such that $Q(z,x) =$ true.

Definition: A protocol (P,V) is a **proof of knowledge for $Q(\cdot,\cdot)$** if the following holds:

- Completeness: V accepts when P has as input an x with $Q(z,x) =$ true.
- Soundness: There exists an efficient program (knowledge extractor) K, which can interact with any program P' for which V accepts with noticeable (also called non-negligible) probability, and outputs a valid secret x.

Note: K can rewind P' (restart with same randomness).

2-Extractability

Definition: A three-move protocol (round) with challenge space C is **2-extractable** if from any two triples (t,c,r) and (t,c',r') with $c \neq c'$ accepted by V one can efficiently compute an x with $Q(z,x) =$ true.

Theorem: An interactive protocol consisting of s 2-extractable rounds with uniform challenge from C is a proof of knowledge if $1/|C|^s$ is negligible.

Proof: Knowledge extractor K:

1. Choose randomness for P' and execute the protocol between P' and V.
2. Execute the protocol again (same randomness for P').

3a. If V accepts in both executions, identify the first round with different challenges c and c' (but same t). Use 2-extractability to compute an x, and output it (and stop).

3b. Otherwise, go back to Step 1.

One-Way Group Homomorphisms (OWGH)

Setting: Groups $\langle G,\star\rangle$ and $\langle H,\otimes\rangle$

Definition: A **group homomorphism** is a function f with:

$$f:\ G \to H, \quad f(a\star b) = f(a)\otimes f(b)$$

Notation: We write $[a]$ for $f(a)$, hence

$$\boxed{[\,]:\ G\to H, \quad [a\star b] = [a]\otimes[b]}$$

Examples

- $G = \langle \mathbb{Z}_q, +\rangle$, $H = \langle h\rangle$ with $|H| = q$, $[a] = h^a$:
$$[a+b] = h^{a+b} = h^a\cdot h^b = [a]\cdot[b]$$
- $G = H = \langle \mathbb{Z}_m^*, \cdot\rangle$, $[a] = a^e$:
$$[a\cdot b] = (a\cdot b)^e = a^e\cdot b^e = [a]\cdot[b].$$

PoK of Pre-Image of OWGH – One Round of the Protocol

Setting: Groups G and H, group homomorphism $[\,] : \langle G,\star\rangle \mapsto \langle H,\otimes\rangle$.
Goal: Prove knowledge of a pre-image x of $z \in H$.

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knows $x \in G$ s.t. $[x] = z$		knows $z \in H$
$k \in_R G$, $t = [k]$	$\xrightarrow{\ t\ }$	
	$\xleftarrow{\ c\ }$	$c \in_R \mathcal{C} \subseteq \mathbb{N}$
$r = k \star x^c$	$\xrightarrow{\ r\ }$	$[r] \stackrel{?}{=} t \otimes z^c$

2-Extractability of OWGH PoK

Theorem 3: The protocol round is 2-extractable if
$\exists \ell \in \mathbb{Z}, u \in G$ s.t. (1) $\forall c_1, c_2 \in \mathcal{C}, c_1 \neq c_2 : \gcd(c_1 - c_2, \ell) = 1$
(2) $[u] = z^\ell$

Proof: Given ℓ and u as above and triples (t, c_1, r_1) and (t, c_2, r_2) with $c_1 \neq c_2$ satisfying the verification test, extract x' with $[x'] = z$ as follows:

1. $$\begin{array}{rcl} [r_1] & = & t\otimes z^{c_1} \\ [r_2] & = & t\otimes z^{c_2} \\ \hline [r_2^{-1}\star r_1] & = & z^{c_1-c_2} \end{array}$$
2. Extended Euclidean Algorithm $\Rightarrow$ a, b with $a\ell + b(c_1 - c_2) = 1$
3. $$z = z^1 = z^{a\ell + b(c_1-c_2)} = z^{a\ell}\otimes z^{b(c_1-c_2)}$$
$$= \left(z^\ell\right)^a \otimes \left(z^{c_1-c_2}\right)^b = [u]^a \otimes [r_2^{-1}\star r_1]^b = [\underbrace{u^a \star (r_2^{-1}\star r_1)^b}_{x'}]$$

OWGH PoK for Schnorr and Guillou-Quisquater

Schnorr

- $G = \mathbb{Z}_q$, cyclic group $H = \langle h \rangle$, $|H| = q$ prime
- $[] : G \to H, \quad x \mapsto [x] = h^x$.
- Thm 3: $\ell = q$, $u = 0$: $z^\ell = 1 = [0]$; q prime $\Rightarrow \mathsf{gcd}(c_1 - c_2, \ell) = 1$.

Guillou-Quisquater

- $G = H = \mathbb{Z}_m^*$.
- $[] : G \to H, \quad x \mapsto [x] = x^e$.
- Thm 3: $\ell = e$, $u = z$: $z^\ell = z^e = [z]$; e prime $\Rightarrow \mathsf{gcd}(c_1 - c_2, \ell) = 1$.

Further Examples

- see paper, lecture, and exercise.