Causal Boxes: Quantum Information-Processing Systems Closed under Composition

Christopher Portmann[*1], Christian Matt[†2], Ueli Maurer[‡2],
Renato Renner[§1], and Björn Tackmann[¶3]

[1]Institute for Theoretical Physics, ETH Zurich, 8093 Zürich, Switzerland.
[2]Department of Computer Science, ETH Zurich, 8092 Zürich, Switzerland.
[3]IBM Research – Zurich, 8803 Rüschlikon, Switzerland.

March 22, 2017

Abstract

Complex information-processing systems, for example quantum circuits, cryptographic protocols, or multi-player games, are naturally described as networks composed of more basic information-processing systems. A modular analysis of such systems requires a mathematical model of systems that is closed under composition, i.e., a network of these objects is again an object of the same type. We propose such a model and call the corresponding systems *causal boxes*.

Causal boxes capture superpositions of causal structures, e.g., messages sent by a causal box A can be in a superposition of different orders or in a superposition of being sent to box B and box C. Furthermore, causal boxes can model systems whose behavior depends on time. By instantiating the Abstract Cryptography framework with causal boxes, we obtain the first composable security framework that can handle arbitrary quantum protocols and relativistic protocols.

---

[*]chportma@ethz.ch

[†]mattc@inf.ethz.ch

[‡]maurer@inf.ethz.ch

[§]renner@ethz.ch

[¶]bta@zurich.ibm.com

Contents

1 Introduction

In this work we are concerned with modeling quantum information-processing systems, i.e., interactive systems that receive and send quantum messages. Similar formalisms for modeling such systems were developed by Gutoski and Watrous [1,2], Chiribella, D'Ariano and Perinotti [3], and Hardy [4–6] (see also Hardy [7,8]), to which we refer in the following using the term from [3], namely *quantum combs*. Quantum combs are a generalization of *random systems* [9,10] to quantum information theory. A comb is a system with internal memory that processes messages one at a time as they are received. An example is depicted in Figure 1, where the dashed lines capture the comb structure.
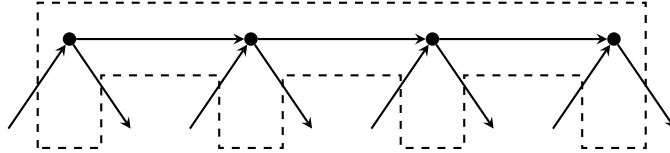
Figure 1 – A single information-processing system modeled as a comb. The nodes represent an operation and the arrows capture a quantum state. Each tooth of the comb corresponds to a pair of an input and an output message.

The quantum comb framework [1–6] provides rules for representing these objects independently from their internal state and composing them when the order of messages is predefined. It also defines a notion of distance between combs. This has found applications in modeling two player games [1] and two player cryptographic protocols [11].

1.1 Ordering messages

The composition of systems described as combs is however not always well-defined. Consider the example drawn in Figure 2: two players, Alice and Bob, each send a message to a third player, Charlie, who outputs the first message he receives and ignores the second. Each of the systems is a well-defined comb. Alice and Bob just output a single message. When Charlie receives the first message, $m = (v, p)$ — value v from player p — he outputs v and ignores all further inputs. But the composition of all three systems (depicted as a dashed box in Figure 2) is not defined: it is a system with no input and one output, but this output is undetermined.

The composition of these three systems is undefined, because Alice's and Bob's messages are unordered, yet the output of Charlie depends on this order. However, if one considers physical systems, e.g., an implementation of Figure 2, the composition is a new well-defined physical system. Messages are output at a certain time, which results in a well-defined order. This ordering information was ignored in the descriptions of the systems given
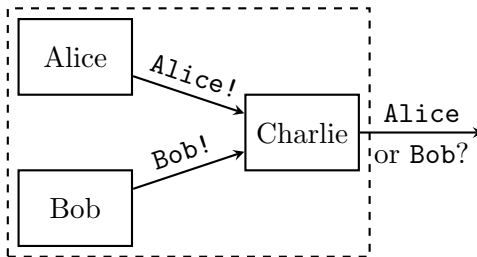
Figure 2 – Alice and Bob both send messages to Charlie, who outputs the first message he receives. Although each system can be described by a comb, the composition of the three, depicted as the dashed box, is undefined.

above, and results in the ill-defined composition. Combs are well-suited for analysing systems that have a predefined causal structure, e.g., ordered networks [3], some simple two player games [1] and two player cryptographic protocols [11]. But Chiribella et al. [12] prove that their framework is ill-suited for modeling settings where the causal structure is not predefined, e.g., when it is determined by an input or a coin toss.[1]

1.2   Superpositions of orders

In a quantum framework it is not sufficient to have all messages (dynamically) ordered, it must also be possible to have messages in superpositions of different orders. For example, a player might choose to send a message $|\psi\rangle$, she might choose to send "nothing" — which we denote by a vacuum state $|\Omega\rangle$ — or she might send a superposition of the two, i.e., she prepares and sends the state

$$\alpha|\psi\rangle + \beta|\Omega\rangle\,. \tag{1}$$

A message could also be in a superposition of sent to Alice and sent to Bob, i.e.,

$$\alpha|\psi\rangle_A \otimes |\Omega\rangle_B + \beta|\Omega\rangle_A \otimes |\psi\rangle_B\,. \tag{2}$$

This results in a player receiving superpositions of different numbers of messages in different orders. Since the value of a message can only be influenced by a message that is ordered before, a superposition of different orders of messages corresponds to a superposition of the causal structure between these messages.

Such superpositions are not only a possibility offered by quantum mechanics, but are also necessary to perform certain quantum information-processing tasks. Consider the problem of designing a circuit that performs a controlled unitary for an unknown U given only a single black-box access to U, which we illustrate in Figure 3a. It has been proven in [13, 14] that

---

[1]To solve this, Chiribella et al. [12] propose a non-causal model of quantum information-processing systems, which we discuss briefly in Section 1.5.

4

Construction:
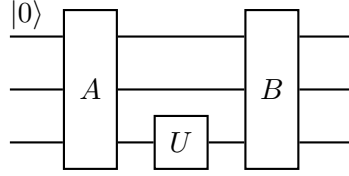
Goal:
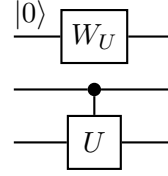
(a) Impossibility of a `ctrl`-U: there do not exist any operators A and B such that for all U the circuit on the left is equivalent to the circuit on the right [13, 14].
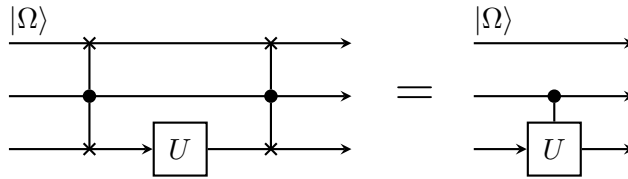
(b) Implementation of a `ctrl`-U: the circuit on the left performs a controlled switch between the top and bottom wires before and after the unitary U, which is equivalent to the controlled unitary depicted on the right [14–16]. When nothing — i.e., the vacuum state $|\Omega\rangle$ — is input to U, nothing is output either.

Figure 3 – (Im)Possibility of implementing a `ctrl`-U. In the circuits in (a) a wire — depicted as a line — corresponds to the space on which U acts. In the circuits in (b) a wire — depicted as an arrow — can also transmit a vacuum state, $|\Omega\rangle$.

this task is impossible. And yet, adding control to an unknown unitary can easily be implemented in practice, and has been done in [15]. As pointed out in [14, 16], the discrepancy between the two results stems from the assumption made in the impossibility proofs [13, 14] that a wire in a circuit models a (non-vacuum) state, whereas in a physical model one has the freedom of not inputing anything to the black box — i.e., one does not use the input wire — in which case no output is produced either. The solution from [15] consists in sending a vacuum state on one of the wires and performing a controlled switch between the vacuum state and the input. This is illustrated in Figure 3b, where the wires are depicted as arrows to emphasize that they have a different meaning from the wires in Figure 3a. We refer to [14, 16] for a further discussion of this.

Another example is given by the *quantum switch* [12]: using only a single black-box access to each of two systems implementing some unknown unitaries U and V, the quantum switch applies these gates in (a controlled) superposition of the two possible orders, i.e., on input

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |\varphi\rangle, \tag{3}$$
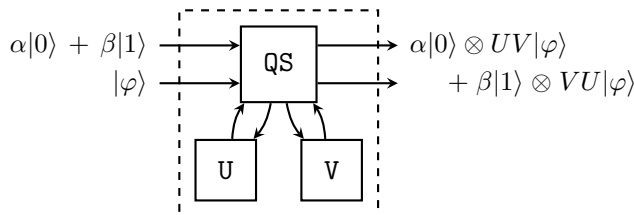
Figure 4 – The quantum switch, `QS`, queries first `U` then `V` or first `V` then `U` depending on the control qubit.

it outputs

$$\alpha|0\rangle \otimes UV|\varphi\rangle + \beta|1\rangle \otimes VU|\varphi\rangle, \tag{4}$$

as illustrated in Figure 4. This particular system — which can be physically implemented [16–18], but cannot be expressed using a classical ordering of gates [12] — has been proven to be useful in reducing the complexity of computational tasks [17,19] and increasing the success probability of discriminating between quantum states and channels [20,21]. As an illustration of our framework we show in Section 8 how to describe the quantum switch using the language of causal boxes that we develop in this work.

It is thus essential that a framework for modeling quantum information processes can capture superpositions of different numbers of messages arriving in superpositions of different orders. To the best of our knowledge, no existing causal framework has such a feature. Physical systems such as those from Figure 3b and Figure 4, and constructions from [14–21] cannot be modeled in existing frameworks. Furthermore, impossibility results proven for combs [12] or a restricted circuit model [13,14] do not hold anymore when we consider the larger class of systems allowed by quantum mechanics. Similarly, existing quantum composable security frameworks [22,23] can only make security statements for a limited class of quantum information-processing systems. They include an artificial classical scheduler that systematically measures states such as (1) and (2) to determine who is the recipient of the next message, and can thus only model a classical ordering of messages.[2]

1.3 Causal boxes

The main contribution of this work is to introduce *causal boxes* as an abstract model for discrete quantum information-processing systems. Causal boxes include quantum combs [1–6] and the models from composable security frameworks [22–24] as special cases. Crucially, they have the property of being closed under composition. Naturally, composing two physical systems

---

[2]The composable security framework from [24] has an additional restriction on the quantum systems that can be modeled in the framework: they must correspond to a set of ordered gates, which is essentially equivalent to an ordered network captured by a quantum comb [1–6].
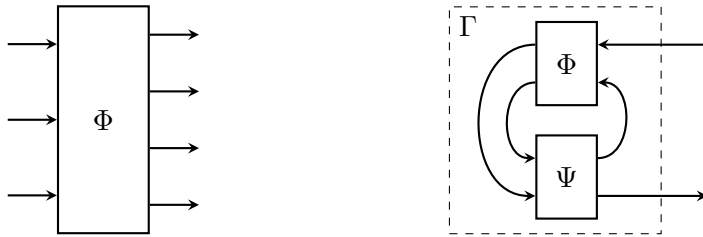
(a) An abstract depiction of a causal box with a flat interface. Messages may arrive in any order — or superpositions thereof — on the wires.

(b) The two causal boxes Φ and Ψ are connected by their wires. This results in a new box Γ.

Figure 5 – The input wires of causal boxes are unordered. Instead, the messages traveling on the wires are ordered and processed according to this order, regardless of the wire. An output generated in position t can depend on any input that arrived strictly before t. Cycles are permitted in a network of boxes, since a connection between two systems can capture a physical link between an output and input port.

results in a new physical system. The main challenge when modeling an abstraction of physical systems is to identify the minimal characterization that preserves this closure property without excluding any systems.

In a quantum comb [1–6], the teeth of the comb capture the predefined causal structure of the system. Any input on a wire at one of the teeth can influence only outputs on teeth that are further to the right in Figure 1. Since we are interested in systems which do not have such a predefined order, causal boxes have a flat (unordered) interface, depicted in Figure 5a. Instead, the messages themselves are ordered, each one consisting of a pair (v, t) of a value v and a global order $t \in \mathcal{T}$ from some partially ordered set $\mathcal{T}$. The parameter t captures the order of the message with respect to other messages, i.e., its position within $\mathcal{T}$. A causal box processes inputs according to this order: an output in position t can only depend on inputs arriving strictly before t — that is, the box must satisfy *causality*. Having all messages assigned some order $t \in \mathcal{T}$ allows protocols that involve time to be naturally modeled by interpreting t as the time at which a message is sent or received. A partial order on $\mathcal{T}$ may also be used to model space-time in a relativistic setting. These applications are discussed further in Sections 1.4 and 9.

We provide a characterization of causal boxes which is independent of their internal state. For example, a box which accepts only classical inputs and produces only classical outputs would have a classical description, regardless of whether it internally performs quantum computation. We show how to compose two boxes, i.e., how to define the new box resulting from connecting two boxes by their wires. This is illustrated in Figure 5b. A connection between two boxes can be thought of as a physical connection — a wire literally plugs one box into another. Unlike for ordered networks

modeled by combs [1–6], we allow cycles in networks of causal boxes. In particular, if for a single causal box Φ an input on a wire A generates an output on a wire B, it is perfectly legal to put a loop from B to A. This does not create any causality conflicts since messages are ordered and an output can only depend on inputs that arrived before.

A wire — depicted by an arrow in Figure 5 — does not necessarily represent a single message. It is possible to send no message, multiple messages, or any superposition thereof on a single wire, e.g., states such as those of Eqs. (1) and (2) can be generated by a causal box. A message can then also be in a superposition of arriving in position t_1 and position t_2, and a causal box can process this without breaking the superposition.

We also define a notion of distance between causal boxes as the probability that a distinguisher[3] can correctly guess to which of two boxes it is connected. Considering different classes of distinguishers results in different distance measures. For example, computationally bounded or unbounded distinguishers allow for different notions of security to be captured in cryptographic applications.

1.4   Applications

Distributed systems.   Causal boxes allow arbitrary distributed systems to be modeled, e.g., where multiple messages are sent simultaneously to different subsystems and the order of their arrival determines the outcome. The analysis of simple two player protocols can be performed with quantum combs [1,11] as they have a predefined order — first player A sends a message, then player B, then A, etc. But if the protocol involves more parties or if these players access some external ressource, the order of the messages is not predetermined and causal boxes need to be used to model these systems.

No-go theorems which are proven using a restricted computational model that cannot capture every quantum information-processing system — such as those of [12–14] — merely prove that the model is inadequate to accomplish the desired task. To show that a (quantum) information-theoretic task is impossible, a general framework such as the one we propose must be employed.[4] For example, the authors of [16] suggest that if the provider of a black-box unitary wishes to enforce that its system is not used to implement a controlled unitary,[5] this may be achieved by including a photon number counter in the box that breaks the superpositions between the vacuum and a photon. Causal boxes can be used to prove whether such a construction

---

[3]For simplicity, a distinguisher can be thought of as another causal box. Though the exact definition is slightly more powerful, see Section 7.

[4]Since our framework is *causal*, an impossibility proof would naturally only be meaningful if non-causal systems are unphysical; see the discussion in Section 9.

[5]This can be essential in a cryptographic context: the unitary could contain sensitive information in the global phase, which can be accessed if it is used as a controlled unitary.

does indeed forbid using a black-box unitary in a controlled way.

Computation with superposition of causal orders. Although in most distributed protocols one might expect the players to measure whether they receive a message or not — and thus break any superposition between the causal order of events — certain distributed systems cannot be implemented without preserving such a superposition. A system which transforms any unitary into a controlled unitary — which was discussed in Section 1.2 — is one such example. Furthermore, as shown in [17,19–21], there is a computational advantage in implementing systems that preserve such a superposition of causal orders. In those works, the quantum switch (introduced in Section 1.2) was used to illustrate this advantage. It is rather straightforward to describe the effect of plugging the quantum switch into the systems U and V — this corresponds to the dashed box from Figure 4 with input-output behavior given by Eqs. (3) and (4). However, the quantum switch itself, i.e., the box QS in Figure 4, cannot be described as a circuit or comb [12]. Possible implementations of the quantum switch using linear optics have been suggested [16,17] and an experimental realization has been performed [18], which effectively provide an implementation dependent description of the switch. Non-causal descriptions of the quantum switch have also been proposed [12, 25, 26], but as shown by the implementations [16–18], the quantum switch is a causal system, and a non-causal description is thus quite unsatisfactory. To illustrate our framework, we provide in Section 8 an abstract, but mathematical, (implementation independent) description of the quantum switch as a causal box.

Cryptography. Traditional quantum composable security frameworks [22–24] can model only a classical ordering of messages. The composition theorems of these frameworks are de facto limited to this restricted class of systems. The Abstract Cryptography (AC) framework [27] (see [28] for an introduction to quantum AC) models cryptography as a resource theory: a protocol constructs a resource (e.g., a secure channel) from another resource (e.g., a secret key). The AC framework treats resources as abstract objects, i.e., it only demands that they have certain properties, but does not prescribe a specific model for them. By instantiating the resources and protocols with causal boxes, one immediately gets a fully quantum composable security framework that can handle superpositions of orders of messages.

To model protocols that involve time, the ordered set $\mathcal{T}$ may be interpreted as the time at which a message is sent or received. Players can then be synchronized by clocks and use timeouts: a player aborts or behaves differently if a message has not been received by time t. This has been used, for example, in authentication protocols to reduce the secret key consumption [29, 30]. Relativistic cryptography uses the fact that information cannot

9

be transmitted faster than the speed of light to implement tasks such as bit commitment [31–35] and position verification [36, 37]. However, due to the lack of cryptography frameworks that can handle time, these works use ad hoc security definitions.[6] A partial order on $\mathcal{T}$ captures relativistic space-time in a natural way. This allows relativistic protocols to be modeled in our framework. AC instantiated with causal boxes then provides composable security definitions that are applicable to relativistic cryptography. This is discussed further in Section 9.

If a distributed system does not provide information about the order in which messages are received, it can be modeled as a family of systems, where each element in the family corresponds to one possible behavior of this system — which, ultimately, determines the ordering of the messages (this is discussed further in Section 9). Security then has to hold for every element in the family. Explicitly modeling the set of possible behaviors of a system removes the artificial concept of a scheduler that is present in other frameworks [22, 23]. Requiring that a protocol be secure for all behaviors of the underlying systems is equivalent to considering the worst case scheduling and has the advantage of being sound in a model with multiple adversaries.[7] Scenarios involving mutually distrustful dishonest players can be captured this way [27, 38, 39].

1.5 Related work

As already mentioned in this introduction, the quantum comb framework [1–6] can model any quantum information-processing system in which the order of the messages is classical and predefined, e.g., a quantum circuit and many two-player games. The systems models from quantum composable security frameworks [22, 23] allow a dynamical (but still classical) ordering of messages. This is essential for modeling cryptographic protocols, where it might be decided at runtime (depending on an input or coin flip) if a message is sent.

The need for physical models that are not restricted to classical orderings of messages was — to the best of our knowledge — first noticed by Hardy, who developed a framework for probability theories with indefinite causal structures [7, 8, 40, 41]. This motivated Oreshkov, Costa and Brukner [42, 43] to propose a process matrix formalism for modeling quantum systems with indefinite causal structures: quantum theory is assumed to be valid in local laboratories, but no reference is made to any global causal relations between the operations in the different laboratories. Various aspects of the process

---

[6]A framework that models circuits that are located in space-time is proposed in [37], but it only defines security for position-based verification and authentication. The issue of defining a generic (composable) security framework in space-time is not adressed in that work.

[7]The scheduler is typically controlled by the adversary [22, 23], but if multiple (non-cooperating) adversaries are present, this type of scheduling is ill-defined.

matrix formalism have been developed further in subsequent works, e.g., the multi-party setting [25, 26], extensible causality [25], witnesses for causal non-separability [26, 44], violations of causal inequalities [45–48], classical systems with indefinite causal structures [49, 50], and non-causal circuits [51]. Chiribella et al. [12] develop an equivalent framework in which systems are modeled as higher-order quantum transformations. Oreshkov and Cerf [52] consider an even more general framework: they introduce a model with no predefined time, even within the individual laboratories. This encompasses previous work as special cases.

On one hand, this process matrix formalism and its derivatives are more general than the current work, because they do not only capture superpositions of causal structures, but also non-causal structures, e.g., where a message going from A to B is both (causally) before and after a message from B to A. However, this added generality makes it unsuited for modeling real world applications, e.g., the behavior of players in a cryptographic protocol, since one might be assigning them unphysical behaviors. On the other hand, these models are more restricted than ours, as they do not allow messages to be dynamically ordered, which is essential for many applications. This is discussed further in Section 9.

Concurrently to this work, the same authors developed a theory of deterministic systems [53], which exploits certain properties specific to the classical case. This allows for a framework that is both simpler and more general than what one obtains by restricting the current work to deterministic systems. In particular, the notion of causality from [53] is less restrictive and includes systems that can execute an infinite number of causal steps in a finite amount of time (see Section 4.3 for a discussion of the causality definitions).

1.6   Structure of this paper

In Section 2 we give an overview of an abstract theory of systems. The main body of this work consists in formalizing systems which capture quantum information processes and satisfy this abstract theory — namely causal boxes. In Section 3 we model the state space of a wire. This constitutes the input and output spaces of causal boxes. Then in Section 4 we define causal boxes as maps from the input wires to the output wires. In Section 5 we prove some lemmas on the Stinespring and Choi-Jamiołkowski representations of causal boxes, which serve as a general toolbox in the next sections. We then treat the composition of causal boxes in Section 6, where we define operations for connecting boxes by their wires. In Section 7 we show how to define a notion of distance on the space of causal boxes using distinguishers. In Section 8 we illustrate our framework by using it to model the quantum switch. We conclude in Section 9 with some final remarks and open questions. An overview of the appendices is given on page 44.

2 A theory of systems

Our goal is to model discrete quantum information-processing systems, such that a network of these systems is itself a valid system. In the following we provide a precise formulation of this desideratum. For this we describe a theory of systems which captures the required properties on an abstract level. It serves as a guideline for the main technical contribution of this paper (Sections 3 to 7), which can be understood as instantiating the theory with causal boxes. In this, we follow the top-down paradigm of Maurer and Renner [27], which consists in modeling objects at the highest possible level of abstraction, and then proceeding downwards, introducing in each new lower level only the minimal necessary specializations. An alternative and more detailed formulation of this abstract theory of systems can be found in [53]. A very similar approach has been used by Hardy [54] to model composition of abstract physical objects. Many of the axioms stated in this section and in [53] can also be found in Hardy's work [54], e.g., composition order independence.

On an abstract level an information-processing system is an object that reads inputs and produces outputs. To every such object Φ we assign a set of ports, $\mathsf{ports}(\Phi)$, through which messages are sent and received. Let $\mathfrak{S}$ be the set of all information-processing systems of interest. Two systems can be composed by connecting some of their ports. Let $\Phi, \Psi \in \mathfrak{S}$ and let $P \subseteq \mathsf{ports}(\Phi) \times \mathsf{ports}(\Psi)$ be a set of pairs of compatible ports, i.e., they have the same dimensions, consist of one in- and one out-port and each port appears at most once in P. The composition operation, which we write $\Phi \overset{P}{\longleftrightarrow} \Psi$, must satisfy certain properties.

Closure. The first property needed is *closure under composition*, namely, for any $\Phi, \Psi \in \mathfrak{S}$ and compatible pairs of ports $P \subseteq \mathsf{ports}(\Phi) \times \mathsf{ports}(\Psi)$,

$$\Phi \overset{P}{\longleftrightarrow} \Psi \in \mathfrak{S} \,.$$

In Theorem 6.7 we prove that this holds for causal boxes.

Composition order independence. When drawing systems one typically produces figures as in Figure 6: boxes connected by wires. Such a picture illustrates the fact that the order in which the systems are connected should not matter. This property is called *composition order independence*: for any $\Phi_1, \Phi_2, \Phi_3 \in \mathfrak{S}$ and compatible pairs of ports $P_{ij} \subseteq \mathsf{ports}(\Phi_i) \times \mathsf{ports}(\Phi_j)$,

$$\left(\Phi_1 \overset{P_{12}}{\longleftrightarrow} \Phi_2 \right) \overset{P_{13} \cup P_{23}}{\longleftrightarrow} \Phi_3 = \Phi_1 \overset{P_{12} \cup P_{13}}{\longleftrightarrow} \left(\Phi_2 \overset{P_{23}}{\longleftrightarrow} \Phi_3 \right) . \tag{5}$$

In Theorem 6.11 we prove that Eq. (5) is satisfied by the composition operation on causal boxes.

12
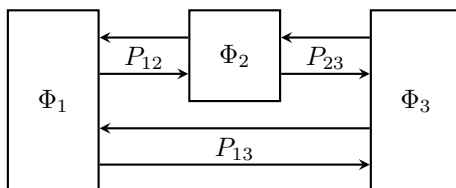
Figure 6 – Systems Φ_1 and Φ_2 are connected by the pairs of ports P_{12}, Φ_1 and Φ_3 are connected by P_{13}, and Φ_2 and Φ_3 are connected by P_{23}. The resulting system must be independent of the order of the connections.

Pseudo-metric. It is often useful for a theory of systems to provide a pseudo-metric on the space of systems $\mathfrak{S}$, namely a function $d : \mathfrak{S} \times \mathfrak{S} \to \mathbb{R}^+$ such that for any $\Phi, \Psi, \Gamma \in \mathfrak{S}$,

$$d(\Phi, \Phi) = 0\,,$$
$$d(\Phi, \Psi) = d(\Psi, \Phi)\,,$$
$$d(\Phi, \Psi) \leq d(\Phi, \Gamma) + d(\Gamma, \Psi)\,.$$

If additionally $d(\Phi, \Psi) = 0 \implies \Phi = \Psi$, then d is a metric.

In Section 7 we define a pseudo-metric on the space of causal systems: the distinguishing advantage. This is defined in terms of the probability that a distinguisher,[8] connected to one of two causal boxes Φ or Ψ, can successfully guess with which causal box it is interacting. We prove in Theorem 7.3 that this is indeed a pseudo-metric — and if the set of all possible distinguishers is considered, the distinguishing advantage is actually a metric.

3 The space of partially ordered messages

3.1 Ordering messages

In a network of information-processing devices, messages are received and sent at a certain time. What is actually received (or sent) by a system can be thought of as a pair (v, t), where v is the message and t the time at which it arrives (or the time at which it is sent). All messages are then naturally ordered. If the input-output behavior of a system is described without any ordering information, one obtains situations as in Figure 2, where the output of a composed system is undefined. However, time captures more information than what is necessary to define an ordering. For information-processing systems to be closed under composition, it is sufficient for every message v to be assigned a position in a partially ordered set. We do this by modeling inputs and outputs of a system as pairs (v, t), where v is a message and

---

[8]Formally, a distinguisher is another causal box that may additional put loops on the system to which it is plugged and outputs a bit corresponding to its guess.

t is an element of some countable,[9] partially ordered set $\mathcal{T}$, denoting the position[10] of v with respect to other messages.

To compose a system Φ in an arbitrary environment, where some part of the environment might run before, after or between any events occurring in the system Φ, one would need to choose $\mathcal{T}$ such that for any $t, t' \in \mathcal{T}$ with $t < t'$, there exist $t_1, t_2, t_3 \in \mathcal{T}$ with $t_1 < t < t_2 < t' < t_3$, e.g., $\mathcal{T} = \mathbb{Q}$. This could be needed for example in a cryptographic setting. For modeling synchronized systems which are divided in rounds, $\mathcal{T} = \mathbb{N}$ might be sufficient. For defining the quantum switch depicted in Figure 4, if we are not modeling composition with any other systems, then it is enough to choose $\mathcal{T} = \{1, \ldots, 6\}$, since exactly 6 messages are produced by these systems (see Section 8 for details, where the quantum switch is modeled as a causal box). In this framework we do not specify the set $\mathcal{T}$ any further than by its countable and partially ordered properties. It is up to each application to define what is needed.

To model quantum messages, we simply define a Hilbert space with a basis given by $\{(v, t)\}_{v,t}$. This is done in Section 3.2. As explained in Section 1.2, the space of a wire should also contain an element representing no message, multiple messages or any superposition thereof. In the classical case this corresponds to the space of all multisets of pairs (v, t), i.e., a wire might transmit nothing $\{\}$, one message $\{(v, t)\}$, two messages $\{(v_1, t_1), (v_2, t_2)\}$, etc. In the quantum case, this corresponds to the bosonic Fock space. We define the exact space of a wire and its main properties in Section 3.3.

Remark 3.1 (Terminology: wires and systems). It is standard in quantum information theory for the word "system" to refer to the label of a Hilbert space, e.g., a (bipartite) system AB is in a state ρ_{AB}. Since in this work the objects of study are not quantum states but objects that receive and send quantum states, we denote these interactive objects as (information-processing) *systems*. We use the terms *wire* and *sub-wire* to label the input and output Hilbert spaces of a system, e.g., a causal box Φ might be a system that maps states from a (bipartite) input wire AB to a (bipartite) output wire CD.

3.2 Single message space

As introduced in Section 3.1, an input to a system is a pair of a message $v \in \mathcal{V}$ and a position $t \in \mathcal{T}$. The corresponding quantum state is an element

---

[9]Since we consider discrete information-processing systems, it is natural that the input and output spaces also be discrete. $\mathcal{T}$ can be seen as an abstraction of some larger (physical) space, that contains extra (possibly uncountably many) points that are not relevant to the discrete systems modeled, since they are never used.

[10]In the following we use "position" to emphasize the relative meaning of the partial order defined on $\mathcal{T}$, though conceptually it may be simpler to think of $t \in \mathcal{T}$ as the time at which a message is sent or received.

of a Hilbert space with an orthonormal basis given by $\{|v, t\rangle\}_{v \in \mathcal{V}, t \in T}$. For a finite $\mathcal{V}$ and infinite $\mathcal{T}$, this Hilbert space corresponds to

$$\mathbb{C}^{|\mathcal{V}|} \otimes \ell^2(\mathcal{T}) \,, \tag{6}$$

where $\ell^2(\mathcal{T}) = \{(x_t)_{t \in \mathcal{T}} : x_t \in \mathbb{C}, \|x\| < \infty\}$ is the sequence space with bounded 2-norm with $\|x\| = \sqrt{\langle x|x\rangle}$ and[11]

$$\langle x|y\rangle = \sum_{t \in \mathcal{T}} \overline{x_t} y_t \,. \tag{7}$$

In the following we refer to a state $|\psi\rangle \in \mathbb{C}^d \otimes \ell^2(\mathcal{T})$ as a qudit with position information. The orthonormal basis $\{|v, t\rangle\}_{v \in \mathcal{V}, t \in \mathcal{T}}$ is given by $|v, t\rangle = |v\rangle \otimes |t\rangle$, where $\{|v\rangle\}_{v \in \mathcal{V}}$ is an orthonormal basis of $\mathbb{C}^{|\mathcal{V}|}$ and $|t\rangle$ is the sequence with a 1 in position $t \in \mathcal{T}$ and all other elements of the sequence are 0.

In the following we write $|v_t\rangle$ instead of $|v, t\rangle$ for the message $|v\rangle$ arriving in position t. For example, the state $\alpha|0_x\rangle + \beta|1_x\rangle$ corresponds to a message in a superposition of $|0\rangle$ and $|1\rangle$ arriving in position x. And the state $\alpha|0_x\rangle + \beta|0_y\rangle$ corresponds to the message $|0\rangle$ in a superposition of arriving in positions x and y.

3.3 Wires

To capture multiple qudits being sent on the same wire to some system, we define the Hilbert space of a wire as a Fock space. For a Hilbert space $\mathcal{H}$, the corresponding bosonic Fock space is given by

$$\mathcal{F}(\mathcal{H}) \coloneqq \bigoplus_{n=0}^{\infty} \vee^n \mathcal{H} \,,$$

where $\vee^n \mathcal{H}$ denotes the symmetric subspace of $\mathcal{H}^{\otimes n}$, and $\mathcal{H}^{\otimes 0}$ is the one dimensional space containing the vacuum state $|\Omega\rangle$. This is explained in more detail in Appendix A. Applying this to the Hilbert space of a qudit with position information from Eq. (6), we get

$$\mathcal{F}\left(\mathbb{C}^d \otimes \ell^2(\mathcal{T})\right) = \bigoplus_{n=0}^{\infty} \vee^n \left(\mathbb{C}^d \otimes \ell^2(\mathcal{T})\right). \tag{8}$$

The orthogonal subspaces $\vee^n\left(\mathbb{C}^d \otimes \ell^2(\mathcal{T})\right)$ for $n \in \mathbb{N}_0$ capture n messages being sent on a wire. The restriction to the symmetric space guarantees that there is no order amongst the qudits other than what might be defined from

---

[11]For the scalar product $\langle x|y\rangle$ to be defined for arbitrary sequences $(x_t)_{t \in \mathcal{T}}$ and $(y_t)_{t \in \mathcal{T}}$, one would need to specify the order of the summation in Eq. (7). But for the set of sequences with bounded 2-norm that define the Hilbert space $\ell^2(\mathcal{T})$, all orders result in the same scalar product, so we omit it.

their state, e.g., their position in $\mathcal{T}$. For example, if a wire contains two copies of a state $|0\rangle$ in position t, it would be in the state $|0_t\rangle \otimes |0_t\rangle$. If it contains two copies of $|0\rangle$ at different positions t_1 and t_2, it would be in the state $\frac{1}{\sqrt{2}}(|0_{t_1}\rangle \otimes |0_{t_2}\rangle + |0_{t_2}\rangle \otimes |0_{t_1}\rangle)$). When modeling concrete systems, one may always restrict the model by adding constraints, e.g., one may consider only systems that produce at most one message or never produce multiple messages at the same position $t \in \mathcal{T}$.

Let A denote a wire that carries d_A-dimensional messages. We write $\mathcal{F}_A^{\mathcal{T}}$ for the corresponding state space, namely

$$\mathcal{F}_A^{\mathcal{T}} := \mathcal{F}\Big(\mathbb{C}^{d_A} \otimes \ell^2(\mathcal{T})\Big).$$

The subscript allows the spaces of different wires to be distinguished, e.g., two wires A and B have joint space $\mathcal{F}_A^{\mathcal{T}} \otimes \mathcal{F}_B^{\mathcal{T}}$, which we also write $\mathcal{F}_{AB}^{\mathcal{T}}$. For any Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$,

$$\mathcal{F}(\mathcal{H}_A) \otimes \mathcal{F}(\mathcal{H}_B) \cong \mathcal{F}(\mathcal{H}_A \oplus \mathcal{H}_B), \tag{9}$$

where the isomorphism preserves the meaning associated with the bases of the Fock spaces, i.e., a tensor product of two vacuum states on the left in Eq. (9) is mapped to a vacuum state on the right, a tensor product of a vacuum state and one message on the left is mapped to a single message with the same value and position on the right, etc. (see Remark A.1 in Appendix A for the exact isomorphism). From Eq. (9) and the fact that

$$\Big(\mathbb{C}^{d_A} \otimes \ell^2(\mathcal{T})\Big) \oplus \Big(\mathbb{C}^{d_B} \otimes \ell^2(\mathcal{T})\Big) \cong \Big(\mathbb{C}^{d_A} \oplus \mathbb{C}^{d_B}\Big) \otimes \ell^2(\mathcal{T}),$$

one can see that $\mathcal{F}_{AB}^{\mathcal{T}}$ also satisfies Eq. (8), i.e., it can be interpreted as the state space of a single wire carrying messages of dimension $d_A + d_B$. The converse also holds: any wire A of messages of dimension d_A can be split in two sub-wires A_1 and A_2 of messages of dimensions $d_{A_1} + d_{A_2} = d_A$. We then have

$$\mathcal{F}_A^{\mathcal{T}} \cong \mathcal{F}_{A_1}^{\mathcal{T}} \otimes \mathcal{F}_{A_2}^{\mathcal{T}}. \tag{10}$$

Thus, a (sub-)wire A_1 — or more precisely, the corresponding message subspace $\mathbb{C}^{d_{A_1}}$ — may be regarded as a way of labeling a subspace of $\mathbb{C}^{d_A}$, which is useful to specify how causal boxes are connected with each other (e.g., messages in the sub-wire corresponding to $\mathbb{C}^{d_{A_1}}$ are connected to one causal box and those in $\mathbb{C}^{d_{A_2}}$ are connected to another, see Section 6). In the following we refer to the dimension of the messages of a wire as the dimension of the wire, e.g., if we say that the wire A has dimension d_A, then $\mathcal{F}_A^{\mathcal{T}} = \mathcal{F}(\mathbb{C}^{d_A} \otimes \ell^2(\mathcal{T}))$.

For any subset $\mathcal{P} \subseteq \mathcal{T}$, we write

$$\mathcal{F}_A^{\mathcal{P}} := \mathcal{F}\Big(\mathbb{C}^{d_A} \otimes \ell^2(\mathcal{P})\Big) \tag{11}$$

for the space of all states occurring at some position in $\mathcal{P}$, e.g.,

$$\mathcal{F}_A^{\leq t} = \mathcal{F}\Big(\mathbb{C}^{d_A} \otimes \ell^2\big(\mathcal{T}^{\leq t}\big)\Big)\,,$$

where $\mathcal{T}^{\leq t} := \{s \in \mathcal{T} : s \leq t\}$. From Eq. (9) and

$$\mathbb{C}^{d_A} \otimes \ell^2(\mathcal{T}) \cong \Big(\mathbb{C}^{d_A} \otimes \ell^2(\mathcal{P})\Big) \oplus \Big(\mathbb{C}^{d_A} \otimes \ell^2\big(\widetilde{\mathcal{P}}\big)\Big)\,,$$

where $\widetilde{\mathcal{P}} := \mathcal{T} \setminus \mathcal{P}$, we have

$$\mathcal{F}_A^{\mathcal{T}} \cong \mathcal{F}_A^{\mathcal{P}} \otimes \mathcal{F}_A^{\widetilde{\mathcal{P}}}\,. \tag{12}$$

This can be interpreted as splitting a wire A into two sub-wires carrying the messages that are in positions in $\mathcal{P}$ and $\widetilde{\mathcal{P}}$, respectively.

A natural embedding of $\mathcal{F}_A^{\mathcal{P}}$ in $\mathcal{F}_A^{\mathcal{T}}$ is obtained by appending "nothing" to $\mathcal{F}_A^{\mathcal{P}}$, i.e.,

$$\mathcal{F}_A^{\mathcal{P}} \cong \mathcal{F}_A^{\mathcal{P}} \otimes |\Omega\rangle_A^{\widetilde{\mathcal{P}}} \subseteq \mathcal{F}_A^{\mathcal{T}}\,, \tag{13}$$

where $|\Omega\rangle_A^{\widetilde{\mathcal{P}}}$ denotes the one dimensional subspace of $\mathcal{F}_A^{\widetilde{\mathcal{P}}}$ that contains the vacuum state. Throughout this work we use $\mathcal{F}_A^{\mathcal{P}}$ to denote both the space defined in Eq. (11) as well as its embedding in $\mathcal{F}_A^{\mathcal{T}}$ given on the right-hand side of Eq. (13).

4 Defining causal boxes

Intuitively, a causal box is a transformation from an input message space to an output message space that satisfies causality. We introduce these two aspects — the transformation and notion of causality — separately. First, in Section 4.1 we formalize the notion of a transformation from the input to the output space as a set of maps. In Section 4.2 we define terminology that provides us with a more convenient representation of these maps. Then in Section 4.3 we introduce the notion of causality that these maps must satisfy in order to be a valid causal box. We put these two aspects together in Section 4.4, where we give the formal definition of a causal box. Finally, in Section 4.5 we define subnormalized causal boxes.

4.1 A set of maps

Let $\mathcal{F}_X^{\mathcal{T}}$ and $\mathcal{F}_Y^{\mathcal{T}}$ denote the Hilbert spaces of an input wire X and output wire Y. And let $\mathfrak{T}(\mathcal{F}_X^{\mathcal{T}})$ and $\mathfrak{T}(\mathcal{F}_Y^{\mathcal{T}})$ be the corresponding sets of trace class operators.[12] A causal box with these input and output wires can be thought

---

[12]$V \in \mathfrak{T}(\mathcal{H})$ if $\|V\|_{\mathrm{tr}} = \sum_i \langle i|\sqrt{V^\dagger V}|i\rangle < \infty$, where $\{|i\rangle\}$ is an orthonormal basis of $\mathcal{H}$. For example, a density operator on a wire A is a non-negative self-adjoint operator $\rho \in \mathfrak{T}(\mathcal{F}_A^{\mathcal{T}})$ with $\mathrm{tr}\,\rho = 1$.

of as a transformation from the message space of X to the message space of Y. But instead of defining it as a map $\Phi : \mathfrak{T}(\mathcal{F}_X^\mathcal{T}) \to \mathfrak{T}(\mathcal{F}_Y^\mathcal{T})$, a causal box is given by a set of mutually consistent[13] maps

$$\Phi = \left\{ \Phi^{\leq t} : \mathfrak{T}(\mathcal{F}_X^\mathcal{T}) \to \mathfrak{T}\left(\mathcal{F}_Y^{\leq t}\right) \right\}_{t \in \mathcal{T}}, \tag{14}$$

where $\mathcal{F}_Y^{\leq t}$ is the subspace of $\mathcal{F}_Y^\mathcal{T}$ that contains only messages in positions $\mathcal{T}^{\leq t} = \{p \in \mathcal{T} : p \leq t\}$. This allows systems to be included that produce an unbounded number of messages on the entire set $\mathcal{T}$. For example, let $\mathcal{T} = \mathbb{N}$ and consider a beacon system that outputs a state $|0\rangle$ at every point $t \in \mathbb{N}$. This is well-defined on every subset $\{1, \ldots, t\}$, but the limit behavior is not, as it would consist in a box that outputs an "infinite tensor product" of $|0\rangle$.

Remark 4.1 (Finite causal boxes). In Appendix C we show how to define the subset of causal boxes that can be represented by a single map $\Phi : \mathfrak{T}(\mathcal{F}_X^\mathcal{T}) \to \mathfrak{T}(\mathcal{F}_Y^\mathcal{T})$ and which is closed under composition. This roughly corresponds to the set of systems that stop processing inputs after some point $t_{\max} \in \mathcal{T}$ — or after some set of unordered points $T_{\max} \subseteq \mathcal{T}$ in the case of a partial order on $\mathcal{T}$.

4.2 Cuts

A map $\Phi^{\leq t}$ from Eq. (14) produces outputs on the subset of positions $\mathcal{T}^{\leq t} \subseteq \mathcal{T}$, which contains all points $p \leq t$. This subset allows the behavior of a system up to a certain position t to be defined. More generally, we are interested in describing the behavior of systems on subsets of positions $\mathcal{C} \subseteq \mathcal{T}$ that are not (necessarily) upper bounded by a single point t, but by a set $\mathcal{P}$, e.g., $\mathcal{C} = \cup_{p \in \mathcal{P}} \mathcal{T}^{\leq p}$. We refer to such a subset of $\mathcal{T}$ as a *cut*, which we illustrate in Figure 7.

Definition 4.2 (Cut). Let $\mathcal{T}$ be a partially ordered set. We define a *cut of* $\mathcal{T}$ as any subset $\mathcal{C} \subseteq \mathcal{T}$ such that

$$\mathcal{C} = \bigcup_{t \in \mathcal{C}} \mathcal{T}^{\leq t},$$

where $\mathcal{T}^{\leq t} = \{p \in \mathcal{T} : p \leq t\}$. We say that a cut $\mathcal{C}$ is *bounded* if there exists a point $t \in \mathcal{T}$ such that $\mathcal{C} \subseteq \mathcal{T}^{\leq t}$. We denote the set of all cuts of $\mathcal{T}$ as $\mathfrak{C}(\mathcal{T})$ and the set of all bounded cuts as $\overline{\mathfrak{C}}(\mathcal{T})$.

Although it is sufficient to describe a causal box by a set of maps as in Eq. (14), we will actually define it as a set of (mutually consistent) maps (that respect causality),

$$\Phi = \left\{ \Phi^\mathcal{C} : \mathfrak{T}(\mathcal{F}_X^\mathcal{T}) \to \mathfrak{T}(\mathcal{F}_Y^\mathcal{C}) \right\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}.$$

---

[13] *Mutually consistent* means that any two maps $\Phi^{\leq t}$ and $\Phi^{\leq u}$ with $t \leq u$ must produce the same output on $\mathcal{T}^{\leq t}$, i.e., $\Phi^{\leq t} = \mathrm{tr}_{\not\leq t} \circ \Phi^{\leq u}$, where $\mathrm{tr}_{\not\leq t}$ traces out messages in positions $\mathcal{T}^{\leq u} \setminus \mathcal{T}^{\leq t}$. This is introduced formally in Section 4.4.
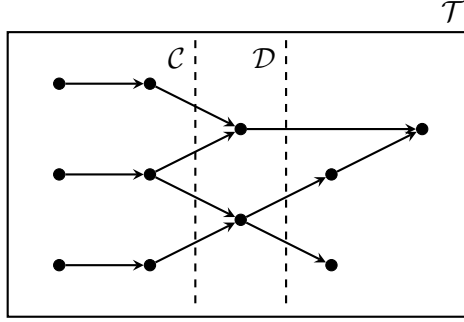
Figure 7 – A partially ordered set $\mathcal{T}$ with two cuts $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{T}$ containing 6 and 8 points, respectively.

These can be derived from Eq. (14) by setting $\Phi^{\mathcal{C}} := \text{tr}_{\widetilde{\mathcal{C}}} \circ \Phi^{\leq t}$, where t is any point such that $\mathcal{C} \subseteq \mathcal{T}^{\leq t}$, $\widetilde{\mathcal{C}} := \mathcal{T}^{\leq t} \setminus \mathcal{C}$ and $\text{tr}_{\widetilde{\mathcal{C}}}$ trace out all messages in positions in $\widetilde{\mathcal{C}}$.

4.3 Causality

To be valid, an information-processing system must respect *causality*: an output can only depend on past inputs. For a totally ordered set $\mathcal{T}$ one can formalize this by requiring that for every causal box there exist a monotone function $\chi : \mathcal{T} \to \mathcal{T}$ such that the output up to position t can be computed from the input up to position $\chi(t) < t$. In the case of a partially ordered $\mathcal{T}$, there might be many unordered points $p < t$ that are needed to compute the output up to position t. We thus define the causality function on cuts, $\chi : \mathfrak{C}(\mathcal{T}) \to \mathfrak{C}(\mathcal{T})$, and require that an output on $\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})$ can be computed from the input on $\chi(\mathcal{C}) \subsetneq \mathcal{C}$.

To be consistent, if the output on $\mathcal{C}$ can be computed from $\chi(\mathcal{C})$ and the output on $\mathcal{D}$ computed from $\chi(\mathcal{D})$, then the output on $\mathcal{C} \cup \mathcal{D}$ can be computed from $\chi(\mathcal{C}) \cup \chi(\mathcal{D})$. So we require $\chi(\mathcal{C} \cup \mathcal{D}) = \chi(\mathcal{C}) \cup \chi(\mathcal{D})$.[14] Furthermore, we also expect that if $\mathcal{C} \subseteq \mathcal{D}$, then $\chi(\mathcal{C}) \subseteq \chi(\mathcal{D})$, because if $\chi(\mathcal{C})$ is needed to compute the output on $\mathcal{C}$, then certainly it is needed for the output on $\mathcal{D} \supseteq \mathcal{C}$.

These conditions are however not sufficient to guarantee that we have well-defined systems. Consider for example a system with $\mathcal{T} = \mathbb{Q}^+$, that for every input received in position $1 - t$, for $0 < t \leq 1$, produces an output in position $1 - t/2$, i.e., $\chi([0, 1 - t/2]) = [0, 1 - t]$. Furthermore, this system initially outputs a message in position 0. If now the messages output are looped back to the input, this system should produce messages at points $\{0, 1/2, 3/4, 7/8, 15/16, \dots\}$. It would effectively output an infinite number

---

[14]This follows immediately if one defines a function $\hat{\chi} : \mathcal{T} \to \mathfrak{C}(\mathcal{T})$ such that the output on $\mathcal{T}^{\leq t}$ can be computed from $\hat{\chi}(t)$ and one sets $\chi(\mathcal{C}) := \bigcup_{t \in \mathcal{C}} \hat{\chi}(t)$.

of messages before position $t = 1$, which is ill-defined. The problem here is that the gap between every input and correlated output gets smaller and smaller as $t \to 1$, and this system executes an infinite number of (causal) steps before reaching $t = 1$. We thus additionally need that every point $t' \in \mathcal{T}$ may be reached from any point $t \le t'$ in a finite number of causal steps. By definition, an output on $\mathcal{T}^{\le t'}$ may be directly computed from an input on $\chi(\mathcal{T}^{\le t'})$. Given an input on $\chi^2(\mathcal{T}^{\le t'})$ on may compute the output on $\mathcal{T}^{\le t'}$ in two steps: one first obtains the output on $\chi(\mathcal{T}^{\le t'})$, then given the input on $\chi(\mathcal{T}^{\le t'})$ one gets the output on $\mathcal{T}^{\le t'}$. In this sense, a point t' is reached from t in a finite number of steps of there exists an n such that $t \notin \chi^n(\mathcal{T}^{\le t'})$, which is captured in Definition 4.3 by Eq. (18).

Definition 4.3 (Causality function). A function $\chi : \mathfrak{C}(\mathcal{T}) \to \mathfrak{C}(\mathcal{T})$ is a *causality function* if it satisfies the following conditions.

$$\forall \mathcal{C}, \mathcal{D} \in \mathfrak{C}(\mathcal{T}), \quad \chi(\mathcal{C} \cup \mathcal{D}) = \chi(\mathcal{C}) \cup \chi(\mathcal{D}), \tag{15}$$

$$\forall \mathcal{C}, \mathcal{D} \in \mathfrak{C}(\mathcal{T}), \quad \mathcal{C} \subseteq \mathcal{D} \implies \chi(\mathcal{C}) \subseteq \chi(\mathcal{D}), \tag{16}$$

$$\forall \mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T}) \setminus \{\emptyset\}, \quad \chi(\mathcal{C}) \subsetneq \mathcal{C}, \tag{17}$$

$$\forall \mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T}), \forall t \in \mathcal{C}, \exists n \in \mathbb{N}, \quad t \notin \chi^n(\mathcal{C}), \tag{18}$$

where χ^n denotes n compositions of χ with itself, $\chi^n = \chi \circ \cdots \circ \chi$.

For example, if $\mathcal{T} = \mathbb{Q}$, a minimum delay δ between every input and correlated output is sufficient to satisfy Definition 4.3. Our definition is thus a generalization of *delta-causality* [55], which requires exactly such a delay δ. Our definition is however more restrictive than the notion of causality used in [53], which simply requires every output to depend only on inputs received strictly earlier, thus allowing an infinite number of outputs to be produced in a finite amount of time.[15]

Remark 4.4 (Causality in totally ordered systems). In the case of a totally ordered $\mathcal{T}$, instead of Eq. (18) one may alternatively require that for $\chi : \mathcal{T} \to \mathcal{T}$ the strict inequality $\chi(t) < t$ must also hold in the limit as $t \to t_0$, i.e., $\inf_{t>t_0} \chi(t) < \inf_{t>t_0} t$ and $\sup_{t<t_0} \chi(t) < \sup_{t<t_0} t$. This is discussed in Appendix D, where we prove that it implies Eq. (18).

4.4 The definition of causal boxes

We can now formally define a causal box.

Definition 4.5 (Causal box). A (d_X, d_Y)-*causal box* Φ is a system with input wire X and output wire Y of dimension d_X and d_Y, defined by a set

---

[15]This is achieved by restricting the systems considered to classical deterministic systems, for which an infinite string of bits is well-defined (as opposed to an infinite tensor product of qubits, which is not).

of mutually consistent, completely positive, trace-preserving (CPTP) maps

$$\Phi = \left\{ \Phi^{\mathcal{C}} : \mathfrak{T}\!\left(\mathcal{F}_X^{\mathcal{T}}\right) \to \mathfrak{T}\!\left(\mathcal{F}_Y^{\mathcal{C}}\right) \right\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})},$$

i.e., for all $\mathcal{C}, \mathcal{D} \in \overline{\mathfrak{C}}(\mathcal{T})$ with $\mathcal{C} \subseteq \mathcal{D}$,

$$\Phi^{\mathcal{C}} = \mathrm{tr}_{\mathcal{D} \setminus \mathcal{C}} \circ \Phi^{\mathcal{D}}, \tag{19}$$

where $\mathrm{tr}_{\mathcal{D} \setminus \mathcal{C}}$ traces out the messages occurring at positions in $\mathcal{D} \setminus \mathcal{C}$.[16] Furthermore, these maps must respect causality: there must exist a function $\chi : \mathfrak{C}(\mathcal{T}) \to \mathfrak{C}(\mathcal{T})$ satisfying Definition 4.3 such that for all $\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})$,[17]

$$\Phi^{\mathcal{C}} = \Phi^{\mathcal{C}} \circ \mathrm{tr}_{\mathcal{T} \setminus \chi(\mathcal{C})}. \tag{20}$$

Eq. (20) says that the output on positions in $\mathcal{C}$ can be computed from the input on positions $\chi(\mathcal{C}) \subsetneq \mathcal{C}$, i.e., the output on $\mathcal{C}$ can only depend on inputs that are before, namely $\chi(\mathcal{C})$. This equation may be alternatively expressed as requiring that $\Phi^{\mathcal{C}} = \hat{\Phi}^{\mathcal{C}} \otimes \mathrm{tr}$, where $\hat{\Phi}^{\mathcal{C}}$ is some map $\mathfrak{T}(\mathcal{F}_X^{\chi(\mathcal{C})}) \to \mathfrak{T}(\mathcal{F}_Y^{\mathcal{C}})$ and tr acts on the space $\mathcal{F}_X^{\mathcal{T} \setminus \chi(\mathcal{C})}$. Using this, Definition 4.5 can be written more compactly as a set of CPTP maps

$$\Phi = \left\{ \Phi^{\mathcal{C}} : \mathfrak{T}\!\left(\mathcal{F}_X^{\chi(\mathcal{C})}\right) \to \mathfrak{T}\!\left(\mathcal{F}_Y^{\mathcal{C}}\right) \right\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})},$$

such that for all $\mathcal{C}, \mathcal{D} \in \overline{\mathfrak{C}}(\mathcal{T})$, $\mathcal{C} \subseteq \mathcal{D}$,

$$\mathrm{tr}_{\mathcal{D} \setminus \mathcal{C}} \circ \Phi^{\mathcal{D}} = \Phi^{\mathcal{C}} \circ \mathrm{tr}_{\mathcal{T} \setminus \chi(\mathcal{C})}. \tag{21}$$

In the following we will often use this form.

Remark 4.6 (Ports). From Eq. (10) we know that a wire can be split into a tensor product of sub-wires. It is thus sufficient to define causal boxes with one index X (Y) for the input (output) wires, as these can be subdivided as needed. For example, Y could consist of two sub-wires Y_1 and Y_2, which are then connected to different systems Ψ and Γ. The ports of an information-processing system, $\mathsf{ports}(\Phi)$, introduced in Section 2 to define composition of systems, correspond to a predefined partition of the input and output wires in sub-wires.

4.5 Subnormalized boxes

Definition 4.5 only considers trace-preserving causal boxes. In Definition 4.7 we generalize this to include trace-decreasing maps. This can be used to model, for example, the subnormalized box resulting from post-selecting states on some measurement outcome.

---

[16] As noted in Eq. (12), $\mathcal{F}_Y^{\mathcal{D}} \cong \mathcal{F}_Y^{\mathcal{C}} \otimes \mathcal{F}_Y^{\mathcal{D} \setminus \mathcal{C}}$.

[17] Eq. (20) uses the embedding of $\mathcal{F}_X^{\chi(\mathcal{C})}$ in $\mathcal{F}_X^{\mathcal{T}}$ so that the output space of $\mathrm{tr}_{\mathcal{T} \setminus \chi(\mathcal{C})}$ matches the input space of $\Phi^{\mathcal{C}}$.

Definition 4.7 (Subnormalized causal box). A set of completely positive (CP) maps,

$$\Phi = \left\{ \Phi^{\mathcal{C}} : \mathfrak{T}\big(\mathcal{F}_X^{\mathcal{T}}\big) \to \mathfrak{T}\big(\mathcal{F}_Y^{\mathcal{C}}\big) \right\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$$

defines a (d_X, d_Y)-*subnormalized causal box* if there exists a normalized causal box

$$\hat{\Phi} = \left\{ \hat{\Phi}^{\mathcal{C}} : \mathfrak{T}\big(\mathcal{F}_X^{\mathcal{T}}\big) \to \mathfrak{T}\big(\mathcal{F}_{RY}^{\mathcal{C}}\big) \right\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$$

such that for all $\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})$,

$$\Phi^{\mathcal{C}} = P_R^{\Omega} \circ \hat{\Phi}^{\mathcal{C}} \,,$$

where $P_R^{\Omega} : \mathfrak{T}(\mathcal{F}_{RY}^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_Y^{\mathcal{T}})$ projects the R sub-wire on the vacuum state, namely

$$P_R^{\Omega}(\rho) \coloneqq (\langle\Omega|_R \otimes I_Y)\rho(|\Omega\rangle_R \otimes I_Y) \,.$$

5 Alternative representations of causal boxes

In this section we provide some alternative characterizations of causal boxes using the Stinespring[18] and Choi-Jamiołkowski[19] representations. These are then used as technical tools in the proofs of the main theorems in the following sections.

5.1 Stinespring and Choi-Jamiołkowski

In Section 4.4 causal boxes are defined as a set of maps $\left\{\Phi^{\mathcal{C}}\right\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ subject to a causality constraint. In this section we show how to represent causal boxes using the Stinespring and Choi-Jamiołkowski representations. The Stinespring representation of a causal box can easily be drawn as a circuit, which we do in Figure 8 — Corollary 5.2 here below states that Figure 8 is indeed equivalent to Eq. (21).

Lemma 5.1. *Let* $\Phi : \mathfrak{T}(\mathcal{H}_{AB}) \to \mathfrak{T}(\mathcal{H}_{CD})$ *be a CPTP map for which the output on* C *does not depend on the input to* B, *i.e., there exists a CPTP map* $\Psi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_C)$ *such that*

$$\mathrm{tr}_D \circ \Phi = \Psi \circ \mathrm{tr}_B \,. \tag{22}$$

The following two conditions are equivalent to Eq. (22).

---

[18]For a detailed treatment of the Stinespring representation of quantum operators on infinite dimensional spaces, we refer to the textbook [56]. A more accessible introduction to the finite dimensional case — which shares many of the essential properties with the infinite dimensional setting — can be found in [57].

[19]Since the Choi-Jamiołkowski representation of infinite dimensional operators differs significantly from the finite dimensional case, we have summarized the most important points in Appendix B.1 and refer the interested reader to [58].
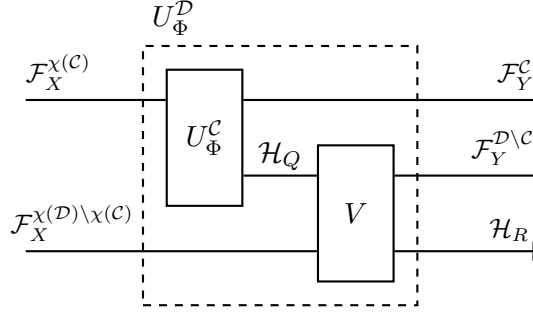
Figure 8 – For any $\mathcal{C}, \mathcal{D} \in \overline{\mathfrak{C}}(\mathcal{T})$ with $\mathcal{C} \subseteq \mathcal{D}$, a map $\Phi^{\mathcal{D}} : \mathfrak{T}(\mathcal{F}_X^{\chi(\mathcal{D})}) \to \mathfrak{T}(\mathcal{F}_Y^{\mathcal{D}})$ of a causal box — with Stinespring representation $U_\Phi^{\mathcal{D}}$ — can be decomposed into a sequence of two isometries $U_\Phi^{\mathcal{C}}$ and V, where $U_\Phi^{\mathcal{C}}$ is a Stinespring representation of $\Phi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_X^{\chi(\mathcal{C})}) \to \mathfrak{T}(\mathcal{F}_Y^{\mathcal{C}})$.

1. *For any states $\psi_A, \varphi_A \in \mathcal{H}_A$, $\psi_B, \varphi_B \in \mathcal{H}_B$, $\psi_C, \varphi_C \in \mathcal{H}_C$, and any basis $\{|j\rangle\}_j$ of $\mathcal{H}_D$,*

$$\sum_j R_\Phi(\psi_C \otimes j_D \otimes \psi_A \otimes \psi_B; \varphi_C \otimes j_D \otimes \varphi_A \otimes \varphi_B)$$

$$= R_\Psi(\psi_C \otimes \psi_A; \varphi_C \otimes \varphi_A)\langle\psi_B|\varphi_B\rangle, \quad (23)$$

   where the sesquilinear forms[20]

$$R_\Phi : (\mathcal{H}_{CD} \times \mathcal{H}_{AB}) \times (\mathcal{H}_{CD} \times \mathcal{H}_{AB}) \to \mathbb{C}$$
$$\text{and} \quad R_\Psi : (\mathcal{H}_C \times \mathcal{H}_A) \times (\mathcal{H}_C \times \mathcal{H}_A) \to \mathbb{C}$$

   are the Choi-Jamiołkowski representations of the maps Φ and Ψ (see Appendix B.1).

2. *There exists an isometry $V : \mathcal{H}_{QB} \to \mathcal{H}_{DR}$ such that*

$$U_\Phi = (I_C \otimes V)(U_\Psi \otimes I_B), \quad (24)$$

   where $U_\Phi : \mathcal{H}_{AB} \to \mathcal{H}_{CDR}$ and $U_\Psi : \mathcal{H}_A \to \mathcal{H}_{CQ}$ are minimal Stinespring representations of Φ and Ψ.

Setting $\mathcal{H}_A = \mathcal{F}_X^{\chi(\mathcal{C})}$, $\mathcal{H}_B = \mathcal{F}_X^{\widetilde{\chi(\mathcal{C})}}$, $\mathcal{H}_C = \mathcal{F}_Y^{\mathcal{C}}$ and $\mathcal{H}_D = \mathcal{F}_Y^{\widetilde{\mathcal{C}}}$ in this lemma, where $\widetilde{\chi(\mathcal{C})} := \chi(\mathcal{D}) \setminus \chi(\mathcal{C})$ and $\widetilde{\mathcal{C}} := \mathcal{D} \setminus \mathcal{C}$, we get the following corollary, which gives an alternative characterization of Eq. (21) and is illustrated in Figure 8.

Corollary 5.2. *Let $\{\Phi_{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_X^{\chi(\mathcal{C})}) \to \mathfrak{T}(\mathcal{F}_Y^{\mathcal{C}})\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ be a set of CPTP maps. For any $\mathcal{C}, \mathcal{D} \in \overline{\mathfrak{C}}(\mathcal{T})$ with $\mathcal{C} \subseteq \mathcal{D}$, the following three conditions are equivalent:*

---

[20]$R(\cdot; \cdot)$ is a sesquilinear form if it is antilinear in the first argument and linear in the second, see Appendix B.1.
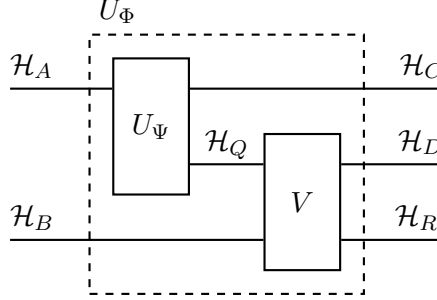
Figure 9 – The Stinespring representation from Lemma 5.1 drawn as a circuit.

1. These maps satisfy Eq. (21), namely

$$\mathrm{tr}_{\widetilde{\mathcal{C}}} \circ \Phi^{\mathcal{D}} = \Phi^{\mathcal{C}} \circ \mathrm{tr}_{\widetilde{\chi(\mathcal{C})}} \ . \tag{25}$$

2. For any states $\psi_X^{\chi(\mathcal{C})}, \varphi_X^{\chi(\mathcal{C})} \in \mathcal{F}_X^{\chi(\mathcal{C})}$, $\psi_X^{\widetilde{\chi(\mathcal{C})}}, \varphi_X^{\widetilde{\chi(\mathcal{C})}} \in \mathcal{F}_X^{\widetilde{\chi(\mathcal{C})}}$, $\psi_Y^{\mathcal{C}}, \varphi_Y^{\mathcal{C}} \in \mathcal{F}_Y^{\mathcal{C}}$, and any basis $\{|j\rangle\}_j$ of $\mathcal{F}_Y^{\widetilde{\mathcal{C}}}$, where $\mathcal{F}_X^{\chi(\mathcal{D})} \cong \mathcal{F}_X^{\chi(\mathcal{C})} \otimes \mathcal{F}_X^{\widetilde{\chi(\mathcal{C})}}$ and $\mathcal{F}_Y^{\mathcal{D}} \cong \mathcal{F}_Y^{\mathcal{C}} \otimes \mathcal{F}_Y^{\widetilde{\mathcal{C}}}$,

$$\sum_j R_\Phi^{\mathcal{D}} \left(\psi_Y^{\mathcal{C}} \otimes j \otimes \psi_X^{\chi(\mathcal{C})} \otimes \psi_X^{\widetilde{\chi(\mathcal{C})}}; \varphi_Y^{\mathcal{C}} \otimes j \otimes \varphi_X^{\chi(\mathcal{C})} \otimes \varphi_X^{\widetilde{\chi(\mathcal{C})}} \right)$$

$$= R_\Phi^{\mathcal{C}} \left(\psi_Y^{\mathcal{C}} \otimes \psi_X^{\chi(\mathcal{C})}; \varphi_Y^{\mathcal{C}} \otimes \varphi_X^{\chi(\mathcal{C})} \right) \left\langle \psi_X^{\widetilde{\chi(\mathcal{C})}} \middle| \varphi_X^{\widetilde{\chi(\mathcal{C})}} \right\rangle, \quad (26)$$

where $R_\Phi^{\mathcal{D}}$ and $R_\Phi^{\mathcal{C}}$ are the Choi-Jamiołkowski representations of $\Phi^{\mathcal{D}}$ and $\Phi^{\mathcal{C}}$.

3. There exists an isometry $V : \mathcal{H}_Q \otimes \mathcal{F}_X^{\widetilde{\chi(\mathcal{C})}} \to \mathcal{F}_Y^{\widetilde{\mathcal{C}}} \otimes \mathcal{H}_R$ such that

$$U_\Phi^{\mathcal{D}} = \left(I_Y^{\mathcal{C}} \otimes V \right) \left(U_\Phi^{\mathcal{C}} \otimes I_X^{\widetilde{\chi(\mathcal{C})}} \right), \tag{27}$$

where $U_\Phi^{\mathcal{D}} : \mathcal{F}_X^{\chi(\mathcal{D})} \to \mathcal{F}_Y^{\mathcal{D}} \otimes \mathcal{H}_R$ and $U_\Phi^{\mathcal{C}} : \mathcal{F}_X^{\chi(\mathcal{C})} \to \mathcal{F}_Y^{\mathcal{C}} \otimes \mathcal{H}_Q$ are minimal Stinespring representations of $\Phi^{\mathcal{D}}$ and $\Phi^{\mathcal{C}}$.

Since the three conditions are equivalent, this corollary not only states that a causal box can be described by the Choi-Jamiołkowski representation of the maps $\Phi^{\mathcal{C}}$, but also that any set of positive semi-definite sesquilinear forms satisfying Eq. (26) is a valid causal box.

We now provide a proof of Lemma 5.1. As visual aid, we draw the corresponding system in Figure 9, which is a copy of Figure 8 with different labels.

Proof of Lemma 5.1. We first show that (22) $\iff$ (23). Note that Eq. (22) is equivalent to requiring that for all states $\psi_A, \psi_B, \psi_C, \varphi_A, \varphi_B, \varphi_C$,

$$\langle \psi_C | \operatorname{tr}_D \big(\Phi \big(|\bar{\psi}_A, \bar{\psi}_B \rangle\langle \bar{\varphi}_A, \bar{\varphi}_B | \big) \big) | \varphi_C \rangle = \langle \psi_C | \Psi \big(\operatorname{tr}_B \big(|\bar{\psi}_A, \bar{\psi}_B \rangle\langle \bar{\varphi}_A, \bar{\varphi}_B | \big) \big) | \varphi_C \rangle \,. \tag{28}$$

The left-hand side of Eq. (28) is equal to

$$\sum_j \langle \psi_C, j_D | \Phi \big(|\bar{\psi}_A, \bar{\psi}_B \rangle\langle \bar{\varphi}_A, \bar{\varphi}_B | \big) | \varphi_C, j_D \rangle \,,$$

which, when rewritten with the Choi-Jamiołkowski representation, corresponds to the left-hand side of Eq. (23). And since

$$\operatorname{tr}_B \big(|\bar{\psi}_A, \bar{\psi}_B \rangle\langle \bar{\varphi}_A, \bar{\varphi}_B | \big) = |\bar{\psi}_A \rangle\langle \bar{\varphi}_A | \langle \psi_B | \varphi_B \rangle \,,$$

the right-hand side of Eq. (28) is equal to the right-hand side of Eq. (23).

Next we show that (24) $\implies$ (22). From the definition of the Stinespring representation and Eq. (24) we have for any $\rho_{AB} \in \mathfrak{T}(\mathcal{H}_{AB})$,

$$\begin{aligned}
\operatorname{tr}_D(\Phi(\rho_{AB})) &= \operatorname{tr}_{DR} \Big((I_C \otimes V)(U_\Psi \otimes I_B)\rho_{AB}(U_\Psi^\dagger \otimes I_B)(I_C \otimes V^\dagger) \Big) \\
&= \operatorname{tr}_{QB} \Big((U_\Psi \otimes I_B)\rho_{AB}(U_\Psi^\dagger \otimes I_B) \Big) \\
&= \operatorname{tr}_Q \Big(U_\Psi \operatorname{tr}_B(\rho_{AB}) U_\Psi^\dagger \Big) \\
&= \Psi(\operatorname{tr}_B(\rho_{AB})) \,,
\end{aligned}$$
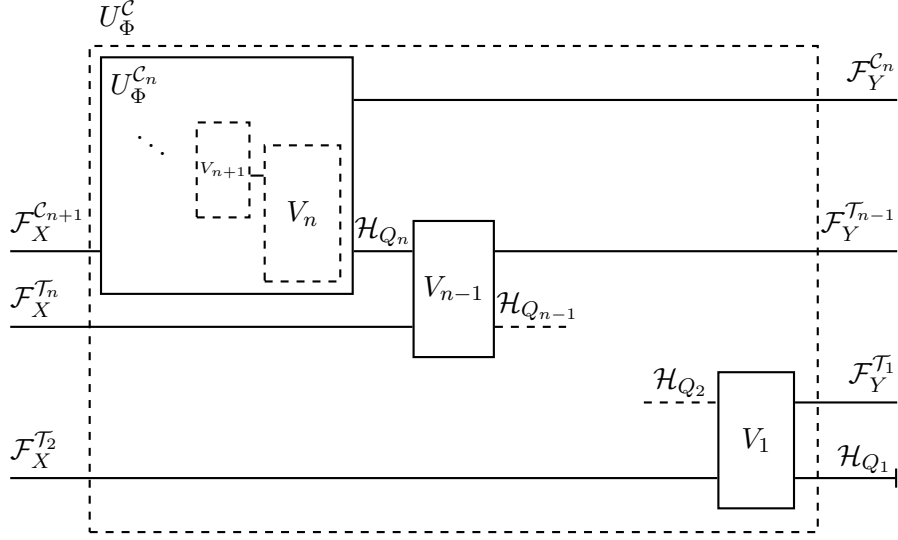
where to obtain the second line we used that since $V : \mathcal{H}_{QB} \to \mathcal{H}_{DR}$ maps QB to DR and the registers DR are traced out, V and $V^\dagger$ have no effect on the outcome (see Figure 9 for an illustration of this case).

Finally, we show that (22) $\implies$ (24). To prove this, we need to find the isometry V which satisfies Eq. (24). Consider the map $\Phi' \coloneqq \operatorname{tr}_D \circ \Phi$. Since $\Phi'(\rho_{AB}) = \operatorname{tr}_{DR}(U_\Phi \rho_{AB} U_\Phi^\dagger)$, the operator U_Φ — which by construction is a Stinespring representation of Φ — is also a Stinespring representation of Φ' with ancilla registers DR. From Eq. (22) we have $\Phi' = \Psi \circ \operatorname{tr}_B$, hence $\Phi'(\rho_{AB}) = \operatorname{tr}_{QB}((U_\Psi \otimes I_B)\rho_{AB}(U_\Psi^\dagger \otimes I_B))$, and $U_\Psi \otimes I_B$ is also a Stinespring representation of Φ', with ancilla QB. Furthermore, since U_Ψ is a minimal representation of Ψ, $U_\Psi \otimes I_B$ is a minimal representation of Φ'. And because any Stinespring representation is related to a minimal one by an isometry on the ancilla [56], there must exist $V : \mathcal{H}_{QB} \to \mathcal{H}_{DR}$ that satisfies Eq. (24). $\square$

5.2   A sequence of operators

If we apply Lemma 5.1 recursively, we can decompose any map $\Phi^{\mathcal{C}}$ into a (finite) sequence of isometries, e.g.,

$$\Big\{ V_i : \mathcal{H}_{Q_{i+1}} \otimes \mathcal{F}_X^{\chi(C_i) \setminus \chi(\mathcal{C}_{i+1})} \to \mathcal{F}_Y^{\mathcal{C}_i \setminus \mathcal{C}_{i+1}} \otimes \mathcal{H}_{Q_i} \Big\}_{i=1}^n \,,$$

Figure 10 – A map $\Phi^{\mathcal{C}}$ is decomposed into a sequence of isometries V_i.

for any sequence of cuts $\emptyset = \mathcal{C}_{n+1} \subseteq \mathcal{C}_n \subseteq \cdots \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_1 = \mathcal{C}$. Here, $\mathcal{H}_{Q_{i+1}}$ can be thought of as the Hilbert space of the internal memory of the system before unitary V_i has been applied, and each V_i processes this internal memory as well as the input on the wire X in positions $\chi(\mathcal{C}_i) \setminus \chi(\mathcal{C}_{i+1})$ to produce an output on the wire Y in positions $\mathcal{C}_i \setminus \mathcal{C}_{i+1}$ and the new updated internal memory $\mathcal{H}_{Q_i}$.

In this section we show that we can do this an infinite number of times in such a way that the input and output sets of positions of the unitaries V_i are disjoint, i.e., $(\chi(C_i) \setminus \chi(\mathcal{C}_{i+1})) \cap (\mathcal{C}_i \setminus \mathcal{C}_{i+1}) = \emptyset$. We call this a *sequence representation* and illustrate it in Figure 10.

Definition 5.3 (Sequence representation)**.** Let $\cdots \subseteq \mathcal{C}_i \subseteq \cdots \subseteq \mathcal{C}_1 = \mathcal{C}$ be an infinite sequence of cuts such that $\bigcap_{i=1}^{\infty} \mathcal{C}_i = \emptyset$, and let $\mathcal{T}_i := \mathcal{C}_i \setminus \mathcal{C}_{i+1}$. A *sequence representation* of a map $\Phi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_X^{\chi(\mathcal{C})}) \to \mathfrak{T}(\mathcal{F}_Y^{\mathcal{C}})$ is given by such a set of cuts $\{\mathcal{C}_i\}_{i=1}^{\infty}$ along with a set of operators

$$\left\{ V_i : \mathcal{H}_{Q_{i+1}} \otimes \mathcal{F}_X^{\mathcal{T}_{i+1}} \to \mathcal{F}_Y^{\mathcal{T}_i} \otimes \mathcal{H}_{Q_i} \right\}_{i=1}^{\infty},$$

such that for all $n \geq 2$,

$$U_{\Phi}^{\mathcal{C}_1} = \left(\prod_{i=1}^{n-1} I_Y^{\mathcal{C}_{i+1}} \otimes V_i \otimes I_X^{\mathcal{C}_2 \setminus \mathcal{C}_{i+1}} \right) \left(U_{\Phi}^{\mathcal{C}_n} \otimes I_X^{\mathcal{C}_2 \setminus \mathcal{C}_{n+1}} \right), \tag{29}$$

where $U_{\Phi}^{\mathcal{C}_i}$ is a minimal Stinespring representation of $\Phi^{\mathcal{C}_i}$.

A sequence representation for any map $\Phi^{\mathcal{C}}$ can be obtained by defining $\mathcal{C}_i := \chi^{i-1}(\mathcal{C})$. We then immediately have that the set of input positions to

the operator V_i has an empty intersection with the output positions, since $\chi(C_i) \setminus \chi(C_{i+1}) = \mathcal{T}_{i+1}$ and $\mathcal{T}_{i+1} \cap \mathcal{T}_i = \emptyset$.

Proposition 5.4. *For every causal box $\Phi = \left\{ \Phi^{\mathcal{C}} \right\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ and every $\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})$, there exists a sequence representation of $\Phi^{\mathcal{C}}$.*

Proof. We fix Φ and $\mathcal{C}$, and define $\mathcal{C}_i := \chi^{i-1}(\mathcal{C})$ and $\mathcal{T}_i := \mathcal{C}_i \setminus \mathcal{C}_{i+1}$. Note that $\mathcal{C}_1 = \mathcal{C}$, $\mathcal{T}_i \cap \mathcal{T}_j = \emptyset$ for $i \neq j$, and from Eq. (18) we have $\bigcap_{i=1}^{\infty} \mathcal{C}_i = \emptyset$. Since by construction of the cuts, $\chi(\mathcal{C}_n) = \mathcal{C}_{n+1}$, the output at all positions $t \in \mathcal{C}_n$ can be computed from the input in positions $\mathcal{C}_{n+1}$, i.e., $\Phi^{\mathcal{C}_n}$ is a map

$$\Phi^{\mathcal{C}_n} : \mathfrak{T}\left(\mathcal{F}_X^{\mathcal{C}_{n+1}} \right) \to \mathfrak{T}\left(\mathcal{F}_Y^{\mathcal{C}_n} \right).$$

Plugging $\Phi^{\mathcal{C}_1}$ and $\Phi^{\mathcal{C}_2}$ in Lemma 5.1, i.e, $\mathcal{H}_A = \mathcal{F}_X^{\mathcal{C}_3}$, $\mathcal{H}_B = \mathcal{F}_X^{\mathcal{T}_2}$, $\mathcal{H}_C = \mathcal{F}_Y^{\mathcal{C}_2}$ and $\mathcal{H}_D = \mathcal{F}_Y^{\mathcal{T}_1}$, we obtain

$$U_{\Phi}^{\mathcal{C}_1} = \left(I_Y^{\mathcal{C}_2} \otimes V_1 \right) \left(U_{\Phi}^{\mathcal{C}_2} \otimes I_X^{\mathcal{T}_2} \right).$$

Repeating this recursively for i going from 2 to n results in the decomposition on the right-hand side of Eq. (29). $\qquad\square$

This proposition can easily be extended to subnormalized causal boxes by appending to every isometry V_i a projector on the vacuum state $|\Omega\rangle$ of the additional wire R. This is illustrated in Figure 11.

Corollary 5.5. *For every subnormalized causal box $\Phi = \left\{ \Phi^{\mathcal{C}} \right\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ and every $\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})$, there exists a sequence representation of $\Phi^{\mathcal{C}}$.*

Proof. Let $\hat{\Phi} = \{ \hat{\Phi}^{\mathcal{C}} : \mathfrak{T}(\mathcal{H}_X) \to \mathfrak{T}(\mathcal{H}_{RY}) \}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ be the corresponding normalized box, and let $\{ \hat{V}_i \}_{i=1}^{\infty}$ be the sequence representation for $\hat{\Phi}^{\mathcal{C}}$. Eq. (29) is satisfied for

$$V_i = \left(\langle \Omega |_R^{\mathcal{T}_i} \otimes I_Y^{\mathcal{T}_i} \otimes I_{Q_i} \right) \hat{U}_i. \qquad\square$$

6 Composing causal boxes

Two causal boxes can be combined by "plugging" output wires into input wires, resulting in a new causal box. Unlike for ordered networks captured by combs [1–6], where cycles are forbidden, here two systems Φ and Ψ can be connected with wires going both from Φ to Ψ and Ψ to Φ, as illustrated in Figure 5b.

Connecting systems can be decomposed in two steps. First Φ and Ψ are composed in parallel, resulting in a new system $\Gamma = \Phi \| \Psi$, whose input and output ports are the union of the ports from Φ and Ψ. Then loops are
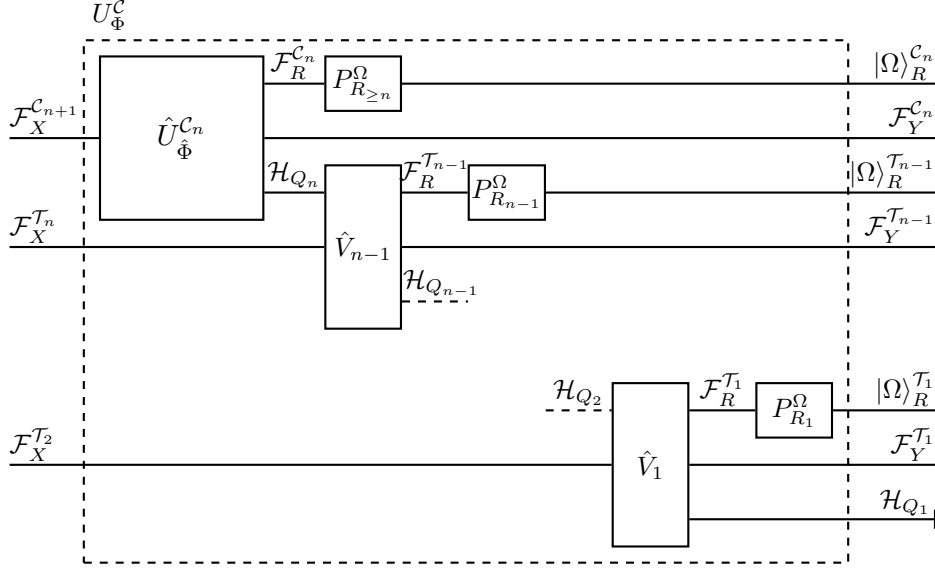
Figure 11 – A trace-decreasing map $\Phi^{\mathcal{C}}$ is decomposed into a sequence of isometries $\hat{V}_i$ followed by a projection $P_{R_i}^{\Omega}$ on the vacuum state of the wire R in the subset $\mathcal{T}_i$.

applied to Γ, connecting its own output and input ports. These two steps are defined in Sections 6.1 and 6.2, respectively.

In Section 6.3 we combine these two steps to define the composition of two systems, $\Phi \overset{P}{\leftrightarrow} \Psi$, introduced in Section 2. We then prove in Theorem 6.7 and Theorem 6.11 that it satisfies closure and composition order independence.

6.1 Parallel composition

We first define parallel composition in Definition 6.1. Then we prove in Proposition 6.2 that the resulting system is still a valid causal box.

Definition 6.1 (Parallel composition)**.** Let $\Phi = \left\{ \Phi^{\mathcal{C}} \right\}_{\mathcal{C} \in \overline{\mathfrak{c}}(\mathcal{T})}$ and $\Psi = \left\{ \Psi^{\mathcal{C}} \right\}_{\mathcal{C} \in \overline{\mathfrak{c}}(\mathcal{T})}$ be (possibly subnormalized) (d_A, d_C)- and (d_B, d_D)-causal boxes. The parallel composition of the two is defined as the $(d_A + d_B, d_C + d_D)$-causal box[21]

$$\Gamma := \left\{ \Phi^{\mathcal{C}} \otimes \Psi^{\mathcal{C}} \right\}_{\mathcal{C} \in \overline{\mathfrak{c}}(\mathcal{T})},$$

which we denote $\Gamma = \Phi \| \Psi$.

---

[21]Recall that the dimension of a wire d_A is the dimension of the messages on the wire, not the dimension of the wire Hilbert space (which is infinite). By Eq. (9) the dimension of the tensor product of two wires is the *sum* of their dimensions.

Proposition 6.2. *If* $\Phi = \{\Phi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_A^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_C^{\mathcal{C}})\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ *and* $\Psi = \{\Psi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_B^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_D^{\mathcal{C}})\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ *are two (sub)normalized causal boxes, then so is* $\Gamma = \Phi \| \Psi$.

Proof. We first consider normalized causal boxes. To prove that $\Phi \| \Psi$ is a valid causal box, we need to find a causality function $\chi_\Gamma : \mathfrak{C}(\mathcal{T}) \to \mathfrak{C}(\mathcal{T})$ such that

$$\Phi^{\mathcal{C}} \otimes \Psi^{\mathcal{C}} = \left(\Phi^{\mathcal{C}} \otimes \Psi^{\mathcal{C}}\right) \circ \mathrm{tr}_{\mathcal{T} \setminus \chi_\Gamma(\mathcal{C})} .$$

We prove in Lemma G.1 that the function $\chi_\Gamma(\mathcal{C}) := \chi_\Phi(\mathcal{C}) \cup \chi_\Psi(\mathcal{C})$ satisfies the requirements of a causality function given in Definition 4.3. And we have

$$\begin{aligned}
\left(\Phi^{\mathcal{C}} \otimes \Psi^{\mathcal{C}}\right) \circ \mathrm{tr}_{\mathcal{T} \setminus \chi_\Gamma(\mathcal{C})} &= \left(\Phi^{\mathcal{C}} \circ \mathrm{tr}_{A^{\mathcal{T} \setminus \chi_\Gamma(\mathcal{C})}}\right) \otimes \left(\Psi^{\mathcal{C}} \circ \mathrm{tr}_{B^{\mathcal{T} \setminus \chi_\Gamma(\mathcal{C})}}\right) \\
&= \left(\Phi^{\mathcal{C}} \circ \mathrm{tr}_{A^{\mathcal{T} \setminus \chi_\Phi(\mathcal{C})}}\right) \otimes \left(\Psi^{\mathcal{C}} \circ \mathrm{tr}_{B^{\mathcal{T} \setminus \chi_\Psi(\mathcal{C})}}\right) \\
&= \Phi^{\mathcal{C}} \otimes \Psi^{\mathcal{C}} ,
\end{aligned}$$

where $\mathrm{tr}_{A^{\mathcal{T} \setminus \chi_\Gamma(\mathcal{C})}}$ and $\mathrm{tr}_{B^{\mathcal{T} \setminus \chi_\Gamma(\mathcal{C})}}$ trace out the inputs in positions $\mathcal{T} \setminus \chi_\Gamma(\mathcal{C})$ on the A and B wires, respectively.

For subnormalized boxes Φ and Ψ, let $\hat{\Phi} = \left\{\hat{\Phi}^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_A^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_{RC}^{\mathcal{C}})\right\}$ and $\hat{\Psi} = \left\{\hat{\Psi}^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_B^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_{SD}^{\mathcal{C}})\right\}$ be their normalized counterparts. Then

$$\begin{aligned}
P_{RS}^{\Omega} \circ \left(\hat{\Phi}^{\mathcal{C}} \otimes \hat{\Psi}^{\mathcal{C}}\right) &= \left(P_R^{\Omega} \circ \hat{\Phi}^{\mathcal{C}}\right) \otimes \left(P_S^{\Omega} \circ \hat{\Psi}^{\mathcal{C}}\right) \\
&= \Phi^{\mathcal{C}} \otimes \Psi^{\mathcal{C}} ,
\end{aligned}$$

where P_R^{Ω} projects the wire R on the vacuum state $|\Omega\rangle$. $\qquad\square$

6.2 Loops

In this section we first give an intuitive explanation of what it means to put a loop from an output wire to an input wire of a causal box and show how to capture this mathematically. We then provide the formal definition in Definition 6.3. In Proposition 6.5 we prove that the resulting system after the loop has been applied is a new valid causal box.

Before considering the case of a loop on an arbitrary causal box, we first look at the simpler case of a map $\Phi : \mathfrak{T}(\mathcal{H}_{AB}) \to \mathfrak{T}(\mathcal{H}_{CD})$ for which the output on C does not depend on the input on B. A Stinespring representation of such a system is illustrated on the left in Figure 12. Here, it is clear what happens when C is looped back to B: first U_Ψ is applied to the input on wire A to obtain the output on C, then V is applied to both C and the internal state of the system to produce the output on D, as depicted on the right in Figure 12.

For the more general case, we make use of Proposition 5.4, which shows that all causal boxes admit a sequence representation, i.e., every map $\Phi^{\mathcal{C}}$

29
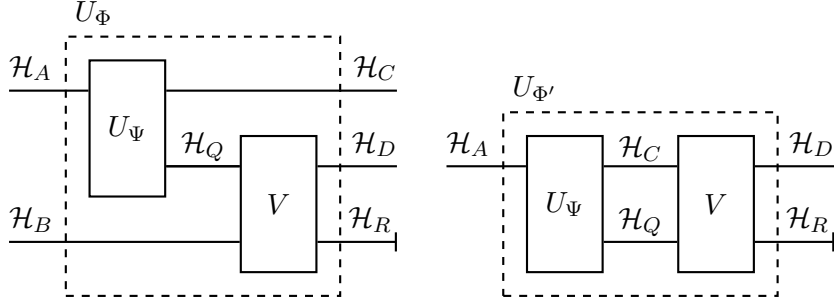
Figure 12 – Connecting C to B in the system depicted on the left results in the system depicted on the right.

can be decomposed into an infinite sequence of isometries whose input and output positions are disjoint. In Figure 13a we reproduce Figure 10, but with two input and output wires. Here, the isometries V_i for $i \geq n$ have been grouped together as one operator $U_\Phi^{\mathcal{C}_n}$ so that $U_\Phi^{\mathcal{C}}$ consists of a finite sequence of operators. We apply the same reasoning as above to this system, i.e., we connect the output on wire C in positions $\mathcal{T}_i$ for $i \leq n$ to the input on wire B of V_{i-1}. This results in a map $\Psi_n^{\mathcal{C}}$, which corresponds to the system $\Phi^{\mathcal{C}}$ with all outputs on wire C in positions $\bigcup_{i=2}^{n} \mathcal{T}_i$ looped back to wire B, and has the Stinespring representation drawn in Figure 13b.

The causal box $\Psi = \Phi^{(C \hookrightarrow B)}$, resulting from looping all of C to B, can be defined as the limit of these maps when $n \to \infty$, i.e., $\Psi := \left\{ \Psi^{\mathcal{C}} \right\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ where

$$\Psi^{\mathcal{C}} := \lim_{n \to \infty} \Psi_n^{\mathcal{C}} \,. \tag{30}$$

This is however a rather inconvenient definition with which to work. Instead, we provide a closed formula for the maps $\{\Psi^{\mathcal{C}}\}_{\mathcal{C}}$ in Definition 6.3, Eq. (31). We prove in Appendix E that this definition is equivalent to Eq. (30).

In the case of a classical causal box given by a set of conditional probability distributions $\{P_{CD|AB}^{\mathcal{C}}\}_{\mathcal{C}}$, this closed formula for a loop reduces to

$$Q_{D|A}^{\mathcal{C}}(d|a) = \sum_c P_{CD|AB}^{\mathcal{C}}(c, d|a, c) \,,$$

i.e., the new system with conditional probability distributions $\{Q_{D|A}^{\mathcal{C}}\}_{\mathcal{C}}$ is obtained from the old system by inputing on wire B the value c that is output on wire C. Eq. (31) is a generalization of this to the quantum case, which uses the Choi-Jamiołkowski representation.[22] In Remark 6.4 we give an equivalent formulation of a loop as a partial trace using the natural representation.[23]

---

[22]The terms in Eq. (31) are absolutely convergent (as proven in Appendix E). The order of the summation is thus not relevant and hence not specified.

[23]We define the natural representation of a map in Appendix B.2.
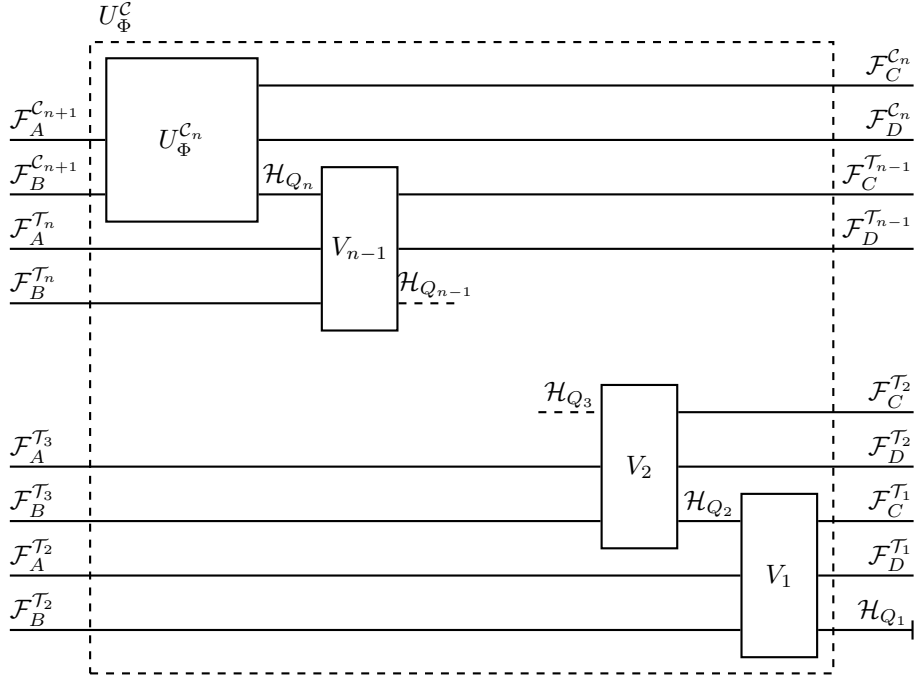
(a) By [Proposition 5.4] any map $\Phi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}^{\mathcal{T}}_{AB}) \to \mathfrak{T}(\mathcal{F}^{\mathcal{C}}_{CD})$ can be decomposed into a sequence of isometries V_i with disjoint input and output positions, as depicted here.
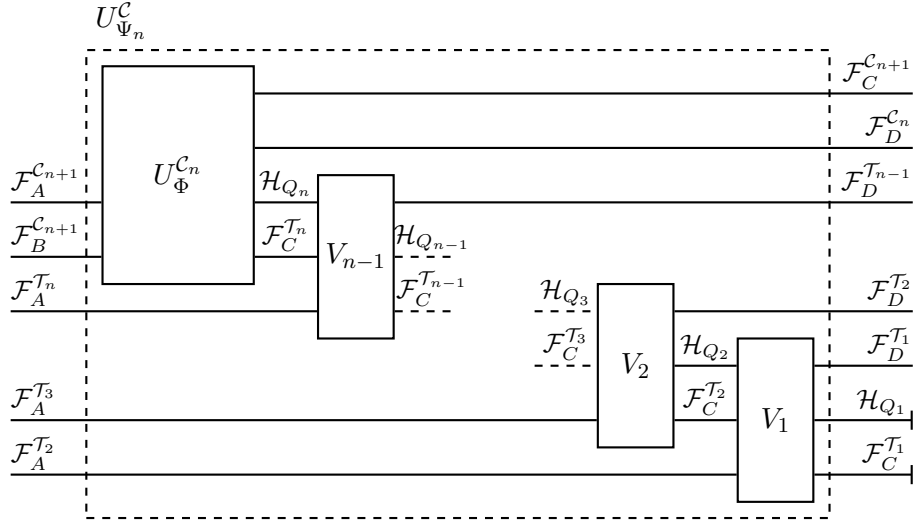
(b) To (partially) connect C to B, the output from V_i in positions $\mathcal{T}_i$ on wire C is input to V_{i+1}.

Figure 13 – Putting a loop from the output wire C to the input wire B in positions $\bigcup_{i=1}^{n} \mathcal{T}_i$ in the system depicted in [Figure 13a] results in the system drawn in [Figure 13b]. Looping all of C back to B corresponds to taking the limit as $n \to \infty$.

31

Definition 6.3 (Loop)**.** Let $\Phi = \{\Phi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_{AB}^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_{CD}^{\mathcal{C}})\}_{\mathcal{C} \in \bar{\mathfrak{C}}(\mathcal{T})}$ be a $(d_A + d_B, d_C + d_D)$-causal box with $d_B = d_C$. Let $R_{\Phi}^{\mathcal{C}}(\cdot; \cdot)$ be the Choi-Jamiołkowski representation of $\Phi^{\mathcal{C}}$. Let $\{|k_C\rangle\}_h$ and $\{|\ell_C\rangle\}_h$ be any orthonormal bases of $\mathcal{F}_C^{\mathcal{C}}$ and let $\{|k_B\rangle\}_k$ and $\{|\ell_B\rangle\}_\ell$ denote the corresponding bases of $\mathcal{F}_B^{\mathcal{C}}$, i.e., for all k and ℓ, $|k_C\rangle \cong |k_B\rangle$ and $|\ell_C\rangle \cong |\ell_B\rangle$. The new system resulting from putting a loop from the output wire C to the input wire B, $\Psi = \Phi^{(C \hookrightarrow B)}$, is given by the set of maps

$$\left\{\Psi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_A^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_D^{\mathcal{C}})\right\}_{\mathcal{C} \in \bar{\mathfrak{C}}(\mathcal{T})}$$

that have Choi-Jamiołkowski representation

$$R_{\Psi}^{\mathcal{C}}(\psi_D \otimes \psi_A; \varphi_D \otimes \varphi_A)$$
$$= \sum_{k,\ell} R_{\Phi}^{\mathcal{C}}(k_C \otimes \psi_D \otimes \psi_A \otimes \bar{k}_B; \ell_C \otimes \varphi_D \otimes \varphi_A \otimes \bar{\ell}_B), \quad (31)$$

where $|\bar{k}_B\rangle = \sum_{i=1}^{\infty} |i\rangle \overline{\langle i|k\rangle}$ for the basis $\{|i_B\rangle\}_i$ of $\mathcal{F}_B^{\mathcal{T}}$ used in the Choi-Jamiołkowski representation of $\Phi^{\mathcal{C}}$.

Remark 6.4 (The natural representation of a loop)*.* In Appendix B.2 we define the natural representation of a map $\Phi : \mathfrak{T}(\mathcal{H}_{AB}) \to \mathfrak{T}(\mathcal{H}_{DC})$ as a linear operator $K_{\Phi} : \mathcal{H}_{A\bar{A}B\bar{B}} \to \mathcal{H}_{D\bar{D}C\bar{C}}$. Relabelling B and $\bar{B}$ with C and $\bar{C}$ according to the isomorphism between the spaces used to define the loop, we can write K_{Φ} as an operator $K_{\Phi} : \mathcal{H}_{A\bar{A}C\bar{C}} \to \mathcal{H}_{D\bar{D}C\bar{C}}$. We show in Lemma B.1 that a map with Choi-Jamiołkowski representation given by Eq. (31) may equivalently be defined by its natural representation

$$K_{\Psi} = \mathrm{tr}_{C\bar{C}}(K_{\Phi}).$$

Since the partial trace is basis independent, the choice of bases in Eq. (31), Definition 6.3, is not relevant.

It follows from Proposition E.2 that if $\Phi^{\mathcal{C}}$ is CPTP, then so is the map $\Psi^{\mathcal{C}}$ with Choi-Jamiołkowski representation given by Eq. (31). We still need to show that the resulting set $\{\Psi^{\mathcal{C}}\}_{\mathcal{C} \in \bar{\mathfrak{C}}(\mathcal{T})}$ is a valid causal box.

Proposition 6.5. *If* $\Phi = \{\Phi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_{AB}^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_{CD}^{\mathcal{C}})\}$ *is a (sub)normalized* $(d_A + d_B, d_C + d_D)$-*causal box with* $d_B = d_C$, *then* $\Psi = \Phi^{(C \hookrightarrow B)}$ *is a (sub)-normalized* (d_A, d_D)-*causal box.*

Proof. We need to show that the maps $\{\Psi^{\mathcal{C}}\}_{\mathcal{C} \in \bar{\mathfrak{C}}(\mathcal{T})}$ satisfy the consistency (Eq. (19)) and causality (Eq. (20)) conditions. We only need to prove this for normalized boxes, since putting a loop on a subnormalized box is equivalent to putting the loop on the underlying normalized box then projecting the ancilla wire R on the vacuum state $|\Omega\rangle_R$, which results in a valid subnormalized box if putting the loop on the underlying normalized box is valid.

To prove this, we use the Choi-Jamiołkowski representation of these conditions, given in Eq. (26). What we need to prove is that there exists a causality function χ such that for any $\mathcal{C}, \mathcal{D} \in \overline{\mathfrak{C}}(\mathcal{T})$ with $\mathcal{C} \subseteq \mathcal{D}$, and any $\psi_D^{\mathcal{C}}$, $\psi_A^{\chi(\mathcal{C})}$, $\psi_A^{\widetilde{\chi(\mathcal{C})}}$, $\varphi_D^{\mathcal{C}}$, $\varphi_A^{\chi(\mathcal{C})}$, $\varphi_A^{\widetilde{\chi(\mathcal{C})}}$,

$$\sum_j R_{\Psi}^{\mathcal{D}}\left(\psi_D^{\mathcal{C}} \otimes j_D^{\widetilde{\mathcal{C}}} \otimes \psi_A^{\chi(\mathcal{C})} \otimes \psi_A^{\widetilde{\chi(\mathcal{C})}}; \varphi_D^{\mathcal{C}} \otimes j_D^{\widetilde{\mathcal{C}}} \otimes \varphi_A^{\chi(\mathcal{C})} \otimes \varphi_A^{\widetilde{\chi(\mathcal{C})}}\right)$$

$$= R_{\Psi}^{\mathcal{C}}\left(\psi_D^{\mathcal{C}} \otimes \psi_A^{\chi(\mathcal{C})}; \varphi_D^{\mathcal{C}} \otimes \varphi_A^{\chi(\mathcal{C})}\right)\left\langle \psi_A^{\widetilde{\chi(\mathcal{C})}} \middle| \varphi_A^{\widetilde{\chi(\mathcal{C})}} \right\rangle, \quad (32)$$

where $R_{\Psi}^{\mathcal{D}}$ and $R_{\Psi}^{\mathcal{C}}$ are the Choi-Jamiołkowski representations of the maps $\Psi^{\mathcal{D}} : \mathfrak{T}(\mathcal{F}_A^{\chi(\mathcal{D})}) \to \mathfrak{T}(\mathcal{F}_D^{\mathcal{D}})$ and $\Psi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_A^{\chi(\mathcal{C})}) \to \mathfrak{T}(\mathcal{F}_D^{\mathcal{C}})$, respectively. We prove this for the same function χ for which Φ satisfies causality.

We denote $\hat{\Psi}^{\mathcal{C}} := \mathrm{tr}_{\widetilde{\mathcal{C}}} \circ \Psi^{\mathcal{D}}$. The left-hand side of Eq. (32) corresponds to the Choi-Jamiołkowski representation of $\hat{\Psi}^{\mathcal{C}}$. Using the loop formula from Eq. (31), we get

$$R_{\hat{\Psi}}^{\mathcal{C}}\left(\psi_D^{\mathcal{C}} \otimes \psi_A^{\chi(\mathcal{D})}; \varphi_D^{\mathcal{C}} \otimes \varphi_A^{\chi(\mathcal{D})}\right)$$

$$= \sum_j R_{\Psi}^{\mathcal{D}}\left(\psi_D^{\mathcal{C}} \otimes j_D^{\widetilde{\mathcal{C}}} \otimes \psi_A^{\chi(\mathcal{D})}; \varphi_D^{\mathcal{C}} \otimes j_D^{\widetilde{\mathcal{C}}} \otimes \varphi_A^{\chi(\mathcal{D})}\right)$$

$$= \sum_{j,k,\ell} R_{\Phi}^{\mathcal{D}}\left(k_C \otimes \psi_D^{\mathcal{C}} \otimes j_D^{\widetilde{\mathcal{C}}} \otimes \psi_A^{\chi(\mathcal{D})} \otimes \bar{k}_B; \ell_C \otimes \varphi_D^{\mathcal{C}} \otimes j_D^{\widetilde{\mathcal{C}}} \otimes \varphi_A^{\chi(\mathcal{D})} \otimes \bar{\ell}_B\right),$$

where $R_{\Phi}^{\mathcal{D}}$ is the Choi-Jamiołkowski representation of $\Phi^{\mathcal{D}} : \mathfrak{T}(\mathcal{F}_{AB}^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_{CD}^{\mathcal{D}})$. Since Φ satisfies causality, the state of the wire B in positions $\mathcal{D} \setminus \mathcal{C}$ cannot modify the output on the wire D in positions $\mathcal{C}$. Hence there is no need to loop all states on C back to B, one could transmit only the states in positions $\mathcal{C}$ and trace out the others without changing the output, i.e.,

$$R_{\hat{\Psi}}^{\mathcal{C}}\left(\psi_D^{\mathcal{C}} \otimes \psi_A^{\chi(\mathcal{D})}; \varphi_D^{\mathcal{C}} \otimes \varphi_A^{\chi(\mathcal{D})}\right) =$$

$$\sum_{i,j,k,\ell} R_{\Phi}^{\mathcal{D}}\left(k_C^{\mathcal{C}} \otimes i_C^{\widetilde{\mathcal{C}}} \otimes \psi_D^{\mathcal{C}} \otimes j_D^{\widetilde{\mathcal{C}}} \otimes \psi_A^{\chi(\mathcal{D})} \otimes \bar{k}_B^{\mathcal{C}};\right.$$

$$\left. \ell_C^{\mathcal{C}} \otimes i_C^{\widetilde{\mathcal{C}}} \otimes \varphi_D^{\mathcal{C}} \otimes j_D^{\widetilde{\mathcal{C}}} \otimes \varphi_A^{\chi(\mathcal{D})} \otimes \bar{\ell}_B^{\mathcal{C}}\right).$$

Applying Eq. (26) we get

$$R_{\hat{\Psi}}^{\mathcal{C}}\left(\psi_D^{\mathcal{C}} \otimes \psi_A^{\chi(\mathcal{D})}; \varphi_D^{\mathcal{C}} \otimes \varphi_A^{\chi(\mathcal{D})}\right) =$$

$$\sum_{i,k,\ell} R_{\Phi}^{\mathcal{C}}\left(k_C^{\mathcal{C}} \otimes \psi_D^{\mathcal{C}} \otimes \psi_A^{\chi(\mathcal{C})} \otimes \bar{k}_B^{\chi(\mathcal{C})}; \ell_C^{\mathcal{C}} \otimes \varphi_D^{\mathcal{C}} \otimes \varphi_A^{\chi(\mathcal{C})} \otimes \bar{\ell}_B^{\chi(\mathcal{C})}\right)$$

$$\left\langle \psi_A^{\widetilde{\chi(\mathcal{C})}} \otimes \bar{k}_B^{\mathcal{C} \setminus \chi(\mathcal{C})} \middle| \varphi_A^{\widetilde{\chi(\mathcal{C})}} \otimes \bar{\ell}_B^{\mathcal{C} \setminus \chi(\mathcal{C})} \right\rangle,$$

where $R_\Phi^\mathcal{C}$ is the Choi-Jamiołkowski representation of $\Phi^\mathcal{C} : \mathfrak{T}(\mathcal{F}_{AB}^{\chi(\mathcal{C})}) \to \mathfrak{T}(\mathcal{F}_{CD}^\mathcal{C})$. Let $\hat{\Phi}^\mathcal{C} := \Phi^\mathcal{C} \otimes \mathrm{tr}$ for a trace operator tr acting on $\mathcal{F}_B^{\mathcal{C}\setminus\chi(\mathcal{C})}$. Then the previous equation may be written

$$R_{\hat{\Psi}}^\mathcal{C}\left(\psi_D^\mathcal{C} \otimes \psi_A^{\chi(\mathcal{D})}; \varphi_D^\mathcal{C} \otimes \varphi_A^{\chi(\mathcal{D})}\right) =$$
$$\sum_{k,\ell} R_\Phi^\mathcal{C}\left(k_C^\mathcal{C} \otimes \psi_D^\mathcal{C} \otimes \psi_A^{\chi(\mathcal{C})} \otimes \bar{k}_B^\mathcal{C}; \ell_C^\mathcal{C} \otimes \varphi_D^\mathcal{C} \otimes \varphi_A^{\chi(\mathcal{C})} \otimes \bar{\ell}_B^\mathcal{C}\right)\left\langle\widetilde{\psi_A^{\chi(\mathcal{C})}}\Big|\widetilde{\varphi_A^{\chi(\mathcal{C})}}\right\rangle.$$

Finally, using Eq. (31), we get

$$R_{\hat{\Psi}}^\mathcal{C}\left(\psi_D^\mathcal{C} \otimes \psi_A^{\chi(\mathcal{D})}; \varphi_D^\mathcal{C} \otimes \varphi_A^{\chi(\mathcal{D})}\right) =$$
$$R_\Psi^\mathcal{C}\left(\psi_D^\mathcal{C} \otimes \psi_A^{\chi(\mathcal{C})}; \varphi_D^\mathcal{C} \otimes \varphi_A^{\chi(\mathcal{C})}\right)\left\langle\widetilde{\psi_A^{\chi(\mathcal{C})}}\Big|\widetilde{\varphi_A^{\chi(\mathcal{C})}}\right\rangle. \quad \square$$

6.3 The composition operation

Now that we have defined parallel composition and loops, we can instantiate the operation for composing two causal boxes by their wires, as introduced in Section 2.

Definition 6.6 (Composition operation)**.** Let Φ and Ψ be two (sub)normalized causal boxes, let $\mathsf{ports}(\Phi)$ and $\mathsf{ports}(\Psi)$ consist of a partition of the input and output wires into sub-wires (see Remark 4.6), and let the set $P = \{(A_1^\Phi, A_1^\Psi), \ldots, (A_n^\Phi, A_n^\Psi)\}$ consist of pairs of ports of Φ and Ψ, such that each pair consists of an output and input sub-wire of the same dimension, and each sub-wire appears at most once. Then

$$\Phi \overset{P}{\longleftrightarrow} \Psi := (\Phi\|\Psi)^{(A_1^{\Phi/\Psi} \hookrightarrow A_1^{\Psi/\Phi})\cdots(A_n^{\Phi/\Psi} \hookrightarrow A_n^{\Psi/\Phi})},$$

where $A_i^{\Phi/\Psi} \hookrightarrow A_i^{\Psi/\Phi}$ denotes either $A_i^\Phi \hookrightarrow A_i^\Psi$ or $A_i^\Psi \hookrightarrow A_i^\Phi$ depending on which is an output and input wire.

Theorem 6.7. *Let $\mathfrak{S}$ be the set of all (sub)normalized causal boxes. For any $\Phi, \Psi \in \mathfrak{S}$ and any set of valid sub-wires P,*

$$\Phi \overset{P}{\longleftrightarrow} \Psi \in \mathfrak{S}.$$

Proof. Immediate from Proposition 6.2 and Proposition 6.5. $\square$

The next step is to prove that Definition 6.6 satisfies composition order independence. To do this, we will prove some lemmas on the commutativity and associativity of parallel composition and loops.

Lemma 6.8. *For any $\Phi, \Psi, \Gamma \in \mathfrak{S}$,*

$$(\Phi\|\Psi)\|\Gamma = \Phi\|(\Psi\|\Gamma).$$

Proof. This follows from the associativity of the tensor product, namely,

$$\left(\Phi^{\mathcal{C}} \otimes \Psi^{\mathcal{C}}\right) \otimes \Gamma^{\mathcal{C}} = \Phi^{\mathcal{C}} \otimes \left(\Psi^{\mathcal{C}} \otimes \Gamma^{\mathcal{C}}\right). \qquad \square$$

Lemma 6.9. *For any* $\Phi, \Psi \in \mathfrak{S}$ *and any pair* $(B,C) \in \mathsf{ports}(\Psi) \times \mathsf{ports}(\Psi)$ *of in- and out-ports of the same dimension,*

$$\Psi^{(C \hookrightarrow B)} \| \Phi = (\Psi \| \Phi)^{(C \hookrightarrow B)},$$
$$\Phi \| \Psi^{(C \hookrightarrow B)} = (\Phi \| \Psi)^{(C \hookrightarrow B)}.$$

Proof. The Choi-Jamiołkowski representation of a product of maps is given by

$$R_{\Phi \otimes \Psi}(\psi_Y \otimes \psi_D \otimes \psi_X \otimes \psi_A; \varphi_Y \otimes \varphi_D \otimes \varphi_X \otimes \varphi_A) =$$
$$R_{\Phi}(\psi_Y \otimes \psi_X; \varphi_Y \otimes \varphi_X) R_{\Psi}(\psi_D \otimes \psi_A; \varphi_D \otimes \varphi_A).$$

Hence for $\Phi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_X^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_Y^{\mathcal{C}})$ and $\Psi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_{AB}^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_{CD}^{\mathcal{C}})$, both $\Phi \| \Psi^{(C \hookrightarrow B)}$ and $(\Phi \| \Psi)^{(C \hookrightarrow B)}$ have Choi-Jamiołkowski representation

$$R_{\Gamma}^{\mathcal{C}}(\psi_Y \otimes \psi_D \otimes \psi_X \otimes \psi_A; \varphi_Y \otimes \varphi_D \otimes \varphi_X \otimes \varphi_A) =$$
$$R_{\Phi}^{\mathcal{C}}(\psi_Y \otimes \psi_X; \varphi_Y \otimes \varphi_X) \sum_{k,\ell} R_{\Psi}^{\mathcal{C}}(k_C \otimes \psi_D \otimes \psi_A \otimes \bar{k}_B; \ell_C \otimes \varphi_D \otimes \varphi_A \otimes \bar{\ell}_B).$$

The proof for $\Psi^{(C \hookrightarrow B)} \| \Phi$ is identical. $\qquad \square$

Lemma 6.10. *For any* $\Phi \in \mathfrak{S}$ *and any pairs* $(A,D), (B,C) \in \mathsf{ports}(\Phi) \times \mathsf{ports}(\Phi)$ *of in- and out-ports of the same dimension,*

$$\left(\Phi^{(C \hookrightarrow B)}\right)^{(D \hookrightarrow A)} = \left(\Phi^{(D \hookrightarrow A)}\right)^{(C \hookrightarrow B)}. \tag{33}$$

Proof. Let $\Phi^{\mathcal{C}} : \mathfrak{T}(\mathcal{F}_{ABX}^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_{CDY}^{\mathcal{C}})$. The Choi-Jamiołkowski representation of the left-hand side of Eq. (33) is

$$R_{\Psi}^{\mathcal{C}}(\psi_Y \otimes \psi_X; \varphi_Y \otimes \varphi_X) =$$
$$\sum_{e,h} \sum_{k,\ell} R_{\Phi}^{\mathcal{C}}(k_C \otimes e_D \otimes \psi_Y \otimes \bar{e}_A \otimes \bar{k}_B \otimes \psi_X; \ell_C \otimes h_D \otimes \varphi_Y \otimes \bar{h}_A \otimes \bar{\ell}_B \otimes \varphi_X).$$

Whereas for the right-hand side we get

$$R_{\Psi}^{\mathcal{C}}(\psi_Y \otimes \psi_X; \varphi_Y \otimes \varphi_X) =$$
$$\sum_{k,\ell} \sum_{e,h} R_{\Phi}^{\mathcal{C}}(k_C \otimes e_D \otimes \psi_Y \otimes \bar{e}_A \otimes \bar{k}_B \otimes \psi_X; \ell_C \otimes h_D \otimes \varphi_Y \otimes \bar{h}_A \otimes \bar{\ell}_B \otimes \varphi_X).$$

The two are equal if the limits implicit in the sums commute. This is the case, because a third expression obtained by first merging the wires A with B and C with D then looping CD to AB results in a valid map

$$\sum_{e,h,k,\ell} R_\Phi^{\mathcal{C}}(k_C \otimes e_D \otimes \psi_Y \otimes \bar{e}_A \otimes \bar{k}_B \otimes \psi_X; \ell_C \otimes h_D \otimes \varphi_Y \otimes \bar{h}_A \otimes \bar{\ell}_B \otimes \varphi_X),$$

and from Lemma G.2 we know that if all three expressions converge, then they must converge to the same value. This can also be seen from the partial trace representation of a loop, namely

$$K_\Psi^{\mathcal{C}} = \mathrm{tr}_{D\bar{D}}\,\mathrm{tr}_{C\bar{C}}\big(K_\Phi^{\mathcal{C}}\big) = \mathrm{tr}_{C\bar{C}D\bar{D}}\big(K_\Phi^{\mathcal{C}}\big) = \mathrm{tr}_{C\bar{C}}\,\mathrm{tr}_{D\bar{D}}\big(K_\Phi^{\mathcal{C}}\big). \qquad \square$$

These three lemmas allow the order in which systems are put in parallel and wires are connected to be changed, which results in composition order independence.

Theorem 6.11. *For any causal boxes $\Phi_1, \Phi_2, \Phi_3 \in \mathfrak{S}$ and compatible pairs of sub-wires $P_{ij} \subseteq \mathsf{ports}(\Phi_i) \times \mathsf{ports}(\Phi_j)$,*

$$\left(\Phi_1 \xleftrightarrow{P_{12}} \Phi_2\right) \xleftrightarrow{P_{13} \cup P_{23}} \Phi_3 = \Phi_1 \xleftrightarrow{P_{12} \cup P_{13}} \left(\Phi_2 \xleftrightarrow{P_{23}} \Phi_3\right).$$

Proof. Immediate by combining Lemmas 6.8, 6.9 and 6.10. $\qquad \square$

7 Distance

In practical applications it is often useful to define a notion of distance between systems, e.g., to measure how close a real system constructed from a noisy channel and an error correcting code is from an ideal channel that has no noise. In this section we introduce a pseudo-metric on the set of causal boxes, the *distinguishing advantage*. This is defined with the help of a distinguisher: another system $\mathcal{D}$ that, when connected to one of two causal boxes, Φ or Ψ, outputs a bit corresponding to its best guess of the system to which it is connected. A distance between these causal boxes is then defined as the statistical distance between the distinguisher's outputs.

Although one could define a distinguisher to be another causal box, we consider a slightly more general definition in this work: a distinguisher may also put loops on the system being tested, i.e., from an output of Φ back to an input of Φ—this is more powerful, because a loop "takes no time", but plugging in a causal box would incur a delay since the identity is not a causal operation. Such a distinguisher is illustrated in Figure 14. Since it only makes sense to compare two systems Φ and Ψ that have inputs and outputs of the same dimensions, we also parametrize the distinguishers by these input and output dimensions.
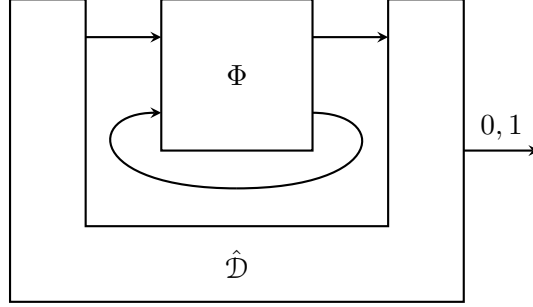
Figure 14 – A distinguisher plugged into a causal box Φ can loop some of the outputs back to the inputs as well as process other outputs before providing them as inputs to Φ. It then outputs a bit corresponding to its decision.

Although one traditionally depicts distinguishers as outputting a bit 0 or 1, we formally model a distinguisher as having a 1-dimensional output wire, i.e., it outputs a state $\rho \in \mathfrak{T}(\mathcal{F}^{\mathcal{T}})$. We then define the distinguisher's decision to be 0 if it outputs a vacuum state, and 1 otherwise. According to our definition of causal boxes, an output only needs to be defined on all bounded cuts $\mathcal{C} \subseteq \mathcal{T}$, it is not necessarily defined on all of $\mathcal{T}$. If the set $\mathcal{T}$ were totally ordered, one could define the output value of the distinguisher to be the limit of the output value on $\mathcal{T}^{\leq t}$ as $t \to +\infty$. However, for a partially ordered set, it is no clear how to take such a limit. Instead, we define the output of a distinguisher $\mathcal{D}$ to correspond to the value produced on a bounded cut $\mathcal{T}^{\leq t_{\mathcal{D}}}$, for some fixed $t_{\mathcal{D}} \in \mathcal{T}$. By considering a set of distinguishers $\mathfrak{D} = \{\mathcal{D}_i\}_i$ that have different bounds $t_{\mathcal{D}_i} \in \mathcal{T}$ and taking the supremum of the distances for each $\mathcal{D}_i \in \mathfrak{D}$, we get a notion of distinguishability that covers all of $\mathcal{T}$.

In Appendix F we discuss some alternative distinguisher definitions. We rewrite the definitions from this section using subnormalized boxes, which simplifies them considerably. Then we consider distinguishers that are not constrained to producing an output within a fixed cut $\mathcal{T}^{\leq t_{\mathcal{D}}}$, and show that this results in an equivalent notion of distance.

Definition 7.1 (Distinguisher)**.** A (m, n)-*distinguisher* $\mathcal{D} = \{\mathrm{id}^{\mathcal{C}} \otimes \hat{\mathcal{D}}^{\mathcal{C}}\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ consists of a $(\hat{n}, \hat{m})$-causal box $\hat{\mathcal{D}} = \{\hat{\mathcal{D}}^{\mathcal{C}}\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ with $m + 1 - \hat{m} = n - \hat{n}$, a termination time $t_{\mathcal{D}} \in \mathcal{T}$, and a specification of how the distinguisher is connected to an (m, n)-dimensional system — i.e., which input and output sub-wires are connected to $\hat{\mathcal{D}}$ and which are directly connected by a loop. We refer to the additional output sub-wire of dimension 1 of the distinguisher that is never connected to any other system as the distinguisher's output wire. For an (m, n)-distinguisher $\mathcal{D}$ and an (m, n)-causal box Φ, let $\mathcal{D}\Phi$ denote the causal box with no input wire and a 1-dimensional output wire resulting from connecting the systems as specified. We define $\mathcal{D}[\Phi]$ to be the binary random variable on $\{0, 1\}$ obtained by projecting the output of $\mathcal{D}\Phi$ within $\mathcal{T}^{\leq t_{\mathcal{D}}}$ on $P_0^{\leq t_{\mathcal{D}}} = |\Omega\rangle\langle\Omega|^{\leq t_{\mathcal{D}}}$ and $P_1^{\leq t_{\mathcal{D}}} = I^{\leq t_{\mathcal{D}}} - |\Omega\rangle\langle\Omega|^{\leq t_{\mathcal{D}}}$.

Definition 7.2 (Distance). Given a set of (m, n)-distinguishers $\mathfrak{D}$, the distance between two (m, n)-causal boxes Φ and Ψ is defined as

$$d^{\mathfrak{D}}(\Phi, \Psi) := \sup_{\mathcal{D} \in \mathfrak{D}} \delta(\mathcal{D}[\Phi], \mathcal{D}[\Psi]),$$

where $\delta(\cdot, \cdot)$ is the statistical or total variation distance, i.e.,

$$\delta(X_1, X_2) = \frac{1}{2} \sum_{x \in \{0,1\}} |\Pr[X_1 = x] - \Pr[X_2 = x]|.$$

We now show that the the function $d^{\mathfrak{D}} : \mathfrak{S} \times \mathfrak{S} \to [0, 1]$ defined above is a pseudo-metric for arbitrary $\mathfrak{D}$ and a metric if $\mathfrak{D}$ is the set of all distinguishers.

Theorem 7.3. *The distance measure given in* Definition 7.2 *is a pseudo-metric. Furthermore, if $\mathfrak{D}$ is the set of all distinguishers, then this distance is a metric.*

Proof. We need to show that the three conditions given in Section 2 hold, namely, for any $\Phi, \Psi, \Gamma \in \mathfrak{S}$,

$$d^{\mathfrak{D}}(\Phi, \Phi) = 0,$$
$$d^{\mathfrak{D}}(\Phi, \Psi) = d(\Psi, \Phi),$$
$$d^{\mathfrak{D}}(\Phi, \Psi) \leq d(\Phi, \Gamma) + d(\Gamma, \Psi).$$

The first two conditions are immediate because the statistical distance is a metric, hence $\delta(\mathcal{D}[\Phi], \mathcal{D}[\Phi]) = 0$ and $\delta(\mathcal{D}[\Phi], \mathcal{D}[\Psi]) = \delta(\mathcal{D}[\Psi], \mathcal{D}[\Phi])$. The last one follows from

$$
\begin{aligned}
d^{\mathfrak{D}}(\Phi, \Psi) &= \sup_{\mathcal{D} \in \mathfrak{D}} \delta(\mathcal{D}[\Phi], \mathcal{D}[\Psi]) \\
&\leq \sup_{\mathcal{D} \in \mathfrak{D}} (\delta(\mathcal{D}[\Phi], \mathcal{D}[\Gamma]) + \delta(\mathcal{D}[\Gamma], \mathcal{D}[\Psi])) \\
&\leq \sup_{\mathcal{D} \in \mathfrak{D}} \delta(\mathcal{D}[\Phi], \mathcal{D}[\Gamma]) + \sup_{\mathcal{D}' \in \mathfrak{D}} \delta(\mathcal{D}'[\Gamma], \mathcal{D}'[\Psi]) \\
&= d^{\mathfrak{D}}(\Phi, \Gamma) + d^{\mathfrak{D}}(\Gamma, \Psi).
\end{aligned}
$$

Finally to show that $d^{\mathfrak{D}}$ is a metric if $\mathfrak{D}$ is the set of all distinguishers, we need to prove that

$$\Phi \neq \Psi \implies \exists \mathcal{D} \in \mathfrak{D} \text{ s.t. } \delta(\mathcal{D}[\Phi], \mathcal{D}[\Psi]) > 0.$$

Since $\Phi \neq \Psi$ there must exist a $\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})$ for which $\Phi^{\mathcal{C}} \neq \Psi^{\mathcal{C}}$, i.e., there exists a $\rho \in \mathfrak{T}(\mathcal{F}_X^{\mathcal{C}_{\text{in}}})$ such that $\Phi^{\mathcal{C}}(\rho) \neq \Psi^{\mathcal{C}}(\rho)$, where $\mathcal{C}_{\text{in}} = \chi_{\Phi}(\mathcal{C}) \cup \chi_{\Psi}(\mathcal{C})$. A distinguisher $\mathcal{D}$ that distinguishes Φ from Ψ can be constructed as the parallel composition of two systems. The first prepares the state ρ and sends it to the system (Φ or Ψ). The second gathers the output of the system until some position t such that $\mathcal{C} \subseteq \mathcal{T}^{\leq t}$. Then it performs an optimal measurement to distinguish $\Phi^{\mathcal{C}}(\rho)$ from $\Psi^{\mathcal{C}}(\rho)$. The result of this measurement is output in a position $t_{\mathcal{D}} \geq t$. $\qquad \square$
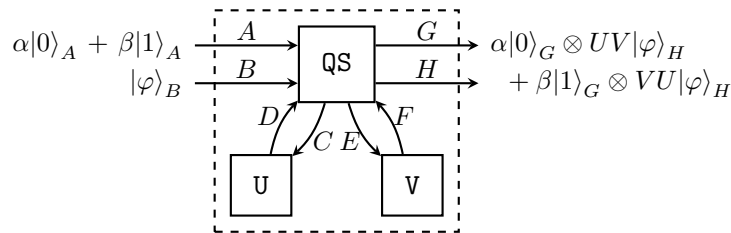
Figure 15 – The quantum switch, QS, queries first U then V or first V then U depending on the control qubit.

8 Example: the quantum switch

The quantum switch — which was introduced in Section 1.2 — is a system that, given black-box access to two unitaries U and V, applies them in a controlled superposition of different orders. We reproduce in Figure 15 a figure illustrating the quantum switch with some extra labels for the different wires.

As shown in [17, 19–21] the quantum switch can decrease the complexity of computational tasks and increase the distinguishability of quantum states and channels. Although it is rather straightforward to describe the effect of plugging the quantum switch into the systems U and V — this corresponds to the dashed box from Figure 15 — the quantum switch itself, i.e., the box QS in Figure 15, cannot be described using combs or circuits [12]. In this section we use our framework to provide a description of the system QS as a causal box.

The total system consists of 3 boxes, U, V and QS. We first model these 3 sub-systems, then prove that the composition of the 3 results in a causal box which applies either UV or VU to an input state according to a control qubit. The entire system can be executed in 6 steps, so it is sufficient to choose $\mathcal{T} = \{1, \dots, 6\}$. The system can easily be extended by adding more points to $\mathcal{T}$ and defining the behavior of the boxes on these points.

The sub-system U applies the corresponding unitary to the input it receives, and outputs the result a step later. For simplicity we assume that if multiple messages are received simultaneously, U is applied to every one of them. We also provide the box with an (internal) counter, that keeps track of the number of times that U is applied. Let $|\psi_t^n\rangle \in \vee^n(\mathbb{C}^d \otimes |t\rangle)$ be an element of the symmetric subspace of n qudits all arriving in position t, for $n \geq 1$. Then

$$\text{U}|\Omega\rangle_C \otimes |i\rangle_\text{U} = |\Omega\rangle_D \otimes |i\rangle_\text{U},$$

$$\text{U}|\psi_t^n\rangle_C \otimes |i\rangle_\text{U} = \left|\left(U^{\otimes n}\psi^n\right)_{t+1}\right\rangle_D \otimes |i + n\rangle_\text{U},$$

where the register C contains the input to U, D contains the output, and the register denoted U is the internal counter of the system. The box V is defined

39

similarly.

As can be seen in Figure 15, the box QS has 8 wires. To provide a complete description of QS we need to define how inputs on the entire Fock space of each input wire are mapped to the output space for all points in $\mathcal{T}$. However, many of these inputs are "invalid", e.g., the initial input is expected to be received in step 1, and QS is plugged into systems U and V that behave as described above. Hence, we describe the behavior of QS on valid inputs only, and assume that any state orthogonal to these are simply ignored by QS.

In the first step, upon receiving the input

$$(\alpha|0\rangle_A + \beta|1\rangle_A) \otimes |\psi\rangle_B\,,$$

QS moves the control qubit to its internal register and queries either U or V conditioned on this state. Let $\mathtt{QS}_1$ denote this operation; it is defined as

$$\mathtt{QS}_1|0\rangle_A \otimes |\psi\rangle_B = |0\rangle_{\mathtt{QS}} \otimes |\Omega\rangle_C \otimes |\psi\rangle_E\,,$$
$$\mathtt{QS}_1|1\rangle_A \otimes |\psi\rangle_B = |1\rangle_{\mathtt{QS}} \otimes |\psi\rangle_C \otimes |\Omega\rangle_E\,.$$

When $t = 3$, QS forwards what it received from U to V and from V to U, i.e.

$$\mathtt{QS}_3|\psi\rangle_D \otimes |\varphi\rangle_F = |\varphi\rangle_C \otimes |\psi\rangle_E\,.$$

And finally, in the last step, conditioned on the value of the control qubit, QS outputs either the message from U or from V along with the control qubit.

$$\mathtt{QS}_5|0\rangle_{\mathtt{QS}} \otimes |\psi\rangle_D \otimes |\Omega\rangle_F = |0\rangle_G \otimes |\psi\rangle_H\,,$$
$$\mathtt{QS}_5|1\rangle_{\mathtt{QS}} \otimes |\Omega\rangle_D \otimes |\psi\rangle_F = |1\rangle_G \otimes |\psi\rangle_H\,.$$

Lemma 8.1. *The composition of* QS, U *and* V *results in a system which performs a controlled switch between the orders of* U *and* V. *Furthermore, the boxes* U *and* V *are queried only once each.*

Proof. To prove this we put together all the steps described above. The wires that do not appear in the equations contain a vacuum state in the corresponding step.

$$t = 1 \qquad (\alpha|0\rangle_A + \beta|1\rangle_A)|\psi\rangle_B|0\rangle_{\mathtt{U}}|0\rangle_{\mathtt{V}}\,,$$
$$t = 2 \qquad \big(\alpha|0\rangle_{\mathtt{QS}}|\Omega\rangle_C|\psi\rangle_E + \beta|1\rangle_{\mathtt{QS}}|\psi\rangle_C|\Omega\rangle_E\big)|0\rangle_{\mathtt{U}}|0\rangle_{\mathtt{V}}\,,$$
$$t = 3 \qquad \alpha|0\rangle_{\mathtt{QS}}|\Omega\rangle_D|V\psi\rangle_F|0\rangle_{\mathtt{U}}|1\rangle_{\mathtt{V}} + \beta|1\rangle_{\mathtt{QS}}|U\psi\rangle_D|\Omega\rangle_F|1\rangle_{\mathtt{U}}|0\rangle_{\mathtt{V}}\,,$$
$$t = 4 \qquad \alpha|0\rangle_{\mathtt{QS}}|V\psi\rangle_C|\Omega\rangle_E|0\rangle_{\mathtt{U}}|1\rangle_{\mathtt{V}} + \beta|1\rangle_{\mathtt{QS}}|\Omega\rangle_C|U\psi\rangle_E|1\rangle_{\mathtt{U}}|0\rangle_{\mathtt{V}}\,,$$
$$t = 5 \qquad \big(\alpha|0\rangle_{\mathtt{QS}}|UV\psi\rangle_D|\Omega\rangle_F + \beta|1\rangle_{\mathtt{QS}}|\Omega\rangle_D|VU\psi\rangle_F\big)|1\rangle_{\mathtt{U}}|1\rangle_{\mathtt{V}}\,,$$
$$t = 6 \qquad (\alpha|0\rangle_G|UV\psi\rangle_H + \beta|1\rangle_G|VU\psi\rangle_H)|1\rangle_{\mathtt{U}}|1\rangle_{\mathtt{V}}\,.$$

In the final step, the wires G and H contain the desired output, and the counters of U and V are set to 1. $\qquad\square$

Remark 8.2 (Alternative quantum switch). The quantum switch defined in this section stores the control qubit in its internal memory. The number of inputs it may treat simultaneously is thus limited by this memory. An alternative construction was proposed independently in [16] and [17] that does not involve any memory: the quantum switch sends the control qubit along with the target qubit to the systems U and V. This allows an unbounded number of switches to be performed simultaneously (and even do them in superposition), but it only works if one can assume that the systems U and V perform their operation on a subspace of the input received and identity on the remaining input.

9 Concluding remarks

Many fundamental notions used to define quantum computation are derived by generalizing classical notions, e.g., quantum Turing machines [59–62], quantum circuits [61,63], or the machine model used in quantum composable security [22,23]. This can result in some classical concepts being hard-coded in the quantum model, e.g., quantum composable security [22,23] can only model a classical scheduling of messages and quantum circuits [61,63] do not consider states in a superposition of being on one wire or another.[24]

In this work we propose a model of quantum systems that is not derived from a classical model, but uses a genuine quantum approach in which the space of quantum messages is a Fock space. The general approach follows the top-down paradigm [27], which consists in starting at the highest possible level of abstraction, and proceeds downwards, introducing in each new lower level only the minimal necessary specializations. The theory of systems introduced in Section 2 does not make any assumptions about the nature of the underlying systems — it applies equally to a classical or quantum model of systems. We instantiate these with causal boxes, which are defined as black boxes. Only the input-output behavior of the system is relevant, the internals — e.g., the memory or model of computation — are not needed and thus not specified. This work leaves open a multitude of questions on modeling discrete quantum systems, some of which we discuss in the following paragraphs.

---

[24]If one interprets the circuit model as forbidding the presence of vacuum states on a wire, then one can prove impossibility results for certain systems which are physically valid [12–14]. However, if one adopts the view introduced in this work that a vacuum state should be modeled explicitly, then one can argue that a message in a superposition between two wires is just an entangled state between two wires, $\alpha|\psi\rangle_A \otimes |\Omega\rangle_B + \beta|\Omega\rangle_A \otimes |\psi\rangle_B$, and that this is captured by the circuit model. And in this sense, the circuit model is (still) a universal model of quantum computation. Nonetheless, the distinction between vacuum and non-vacuum state is vital for computation and query complexity, since if one sends nothing to an oracle, this oracle has not been queried.

Quantum complexity. One of the most fundamental open questions is how to evaluate the complexity of a quantum circuit. A straightforward adaptation of the classical concept of counting gates in a circuit does not seem to be meaningful if two circuits are run in superposition, or if a circuit consists of many gates used in superposition. If one does not distinguish between a gate being used or not (see Footnote 24), then the complexity of the circuit would correspond to counting all the gates needed to draw the entire circuit. In the case of the quantum switch from Section 8, one would find that either U or V has to be queried at least twice [12], even though one of the queries is the vacuum state. The gap between the number of gates needed to draw a circuit and the number of gates actually used can be quite large: Araújo et al. [17] found a problem that cannot be solved with a circuit containing less than $\Omega(n^2)$ gates, but only superpositions of $O(n)$ subsets of these gates are ever used. Note that to implement the algorithm of Araújo et al., it is sufficient to construct $O(n)$ gates, since the circuit contains only $O(n)$ *different* gates and the quantum switch can be used to swap the order of these gates [17]. However, such a coherent counting method cannot be applied in general, since this could result in a measurement of whether the gate is used or not, which could change the outcome of the computation. We would thus evaluate the complexity of a different circuit altogether.[25] A more physical approach to measuring complexity may be needed, e.g., by evaluating the time or energy needed to perform a computation [64, 65].

Indefinite causal structures. Our framework models messages in superpositions of different orders by explicitly defining a set of positions $\mathcal{T}$ and assigning different positions (or superpositions of positions) to different messages. This allows superpositions of different causal structures to be captured by the framework. A different line of research, inspired by Hardy's work on probability theories with indefinite causal structures [7, 8, 40, 41], has developed a process matrix formalism to model indefinite causal structures [25, 26, 42–50]. As already mentioned in Section 1.5, the process matrix framework does not only capture superpositions of causal structures, but also unphysical causal structures. In fact, it is currently still unclear what systems modeled by the process matrix framework are physically implementable. Additionally, the various players connected to the process matrix are restricted to processing inputs and producing outputs exactly once, preventing any dynamical behavior of the parties, e.g., they cannot change at runtime the number of messages they read or send. This framework is thus not suited for modeling the possible behaviors of players in a multi-player game or

---

[25]The example of Araújo et al. [17] is constructed so that coherently counting which gates are used does not change the computation. This is possible when the counter factors out at the end of the computation. This is also the case in the example of the quantum switch from Section 8, where the internal counters of U and V are in a superposition of $|0\rangle$ and $|1\rangle$ in the middle of the protocol, but factor out at the end.

cryptographic protocol.

It is nonetheless interesting to compare what systems can be modeled by either framework. A system that allows a causal inequality to be violated cannot be captured by causal boxes, since a global notion of order is hard-coded in our framework. Hence causal boxes can only model process matrices that are *extensibly causal* — those that cannot be used to violate a causal inequality [25]. However, no violation of a causal inequality has ever been physically realized. In fact, to date, realizable physical experiments have a global notion of order, and thus causal boxes might well capture the subset of process matrices that can be implemented. It remains open whether this corresponds to extensibly causal processes, or whether there are examples of such processes that cannot be physically realized [66].

Relativistic cryptography. An immediate application of our causal boxes framework is in modeling cryptographic protocols that involve time, e.g., relativistic protocols [31–37]. Here, (honest) parties are positioned in precise locations, and the time taken by messages to travel form one location to another is used to ensure that a dishonest party cannot cheat. To model such a setting one can take $\mathcal{T}$ to be a countable subset of space-time. Two points t and t' are ordered, $t \leq t'$, if t' is in the future light cone of t. All messages are then assigned some point $t \in \mathcal{T}$ corresponding to their location.

Non-deterministic systems. Our framework may also be used to model protocols that do not involve time explicitly. In many settings, a player executes a set of instructions, but the time it takes to do so is not determined by the player. One can think of such a system as being *non-deterministic*. It does not correspond to one system, but to a set of systems, each one capturing one possible behavior. For example, a simple identity channel could be a set of systems which forward the message received with a delay δ, where the set is taken over all possible δ, e.g., id = $\{\mathrm{id}_\delta\}_{\delta \in \mathbb{Q}^+}$. The composition of two such non-deterministic systems, $\Pi = \{\pi_x\}_x$ and $\Psi = \{\psi_y\}_y$, is defined as the set resulting from composing all pairs of elements from each system, $\Pi \overset{P}{\leftrightarrow} \Psi = \{\pi_x \overset{P}{\leftrightarrow} \psi_y\}_{x,y}$. Since any possible element $\pi \in \Pi$ could occur if a protocol Π is executed, we require security to hold for all elements $\pi \in \Pi$. A main challenge in modeling systems this way is to determine what set of behaviors correspond to a simple and intuitive, but incomplete, description of a system as, e.g., a protocol given by pseudo-code or a random system [9,10]. Furthermore, for such non-deterministic systems to be usable, one would need to develop formalism that allows them to be described compactly and which satisfies the axioms from Section 2.

43

Acknowledgments

Appendices

In Appendix A we define a Fock space in more detail. In Appendix B we give a brief overview of representations of quantum operators on infinite dimensional systems. We discuss the Choi-Jamiołkowski representation in Appendix B.1 and the natural representation in Appendix B.2. In Appendix C we define the subset of causal boxes that can be represented as a single map and which is closed under composition. In Appendix D we discuss the special case of the causality function for systems with a totally ordered set $\mathcal{T}$. We prove that in this case, for the composition of two systems to be well-defined, it is sufficient to require that the systems are not arbitrarily fast around any point. In Appendix E we prove some additional results about the loop operation. We show that the alternative definition of a loop from Section 6.2 is equivalent to Definition 6.3. This simultaneously proves that Definition 6.3 always yields normalized maps. In Appendix F we give some alternative distinguisher definitions, and show that they result in the same pseudo-metric as in Section 7. In particular, we show how this distance can be defined with subnormalized distinguishers. Finally, Appendix G contains technical lemmas.

A Fock space

As introduced in Section 3.3, for a Hilbert space $\mathcal{H}$, the corresponding bosonic Fock space is given by

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \vee^n \mathcal{H}, \tag{34}$$

where $\vee^n \mathcal{H}$ denotes the symmetric subspace of $\mathcal{H}^{\otimes n}$, and $\mathcal{H}^{\otimes 0}$ is the one dimensional space containing the vacuum state $|\Omega\rangle$. Let $\{|i\rangle : i \in \mathcal{B}\}$ denote a basis of $\mathcal{H}$, where $\mathcal{B}$ is a strict totally ordered set. For

$$\mathcal{B}^{\vee n} := \{(i_1, \ldots, i_n) : i_j \in \mathcal{B}, j < k \implies i_j \leq i_k\}, \tag{35}$$

a basis of $\vee^n \mathcal{H}$ is then given by $\{|(i_1, \ldots, i_n)\rangle : (i_1, \ldots, i_n) \in \mathcal{B}^{\vee n}\}$ where $|(i_1, \ldots, i_n)\rangle$ denotes the superposition over all different states obtained by permuting the positions in the vector: for $(i_1, \ldots, i_n) \in \mathcal{B}^{\vee n}$, let

$$\mathcal{B}^{\times n}_{(i_1, \ldots, i_n)} := \bigcup_{\pi \in S_n} \{(i_{\pi(1)}, \ldots, i_{\pi(n)})\}$$

denote the subset of $\mathcal{B}^{\times n}$ obtained by permuting the elements of $(i_1, \ldots, i_n)$, then

$$|(i_1, \ldots, i_n)\rangle := \frac{1}{\sqrt{\left|\mathcal{B}^{\times n}_{(i_1, \ldots, i_n)}\right|}} \sum_{(j_1, \ldots, j_n) \in \mathcal{B}^{\times n}_{(i_1, \ldots, i_n)}} |j_1, \ldots, j_n\rangle. \tag{36}$$

And a basis of $\mathcal{F}(\mathcal{H})$ is given by

$$\left\{|x\rangle : x \in \bigcup_{n=0}^{\infty} \mathcal{B}^{\vee n}\right\}.$$

For a wire A with Hilbert space $\mathcal{F}_A^{\mathcal{T}} = \mathcal{F}(\mathbb{C}^{d_A} \otimes \ell^2(\mathcal{T}))$, a state $|\Psi\rangle \in \mathcal{F}_A^{\mathcal{T}}$ can thus be in a superposition consisting of any number of qudits, e.g.,

$$|\Psi\rangle = \sum_{n=0}^{\infty} \alpha_n |\Psi_n\rangle,$$

where $|\Psi_n\rangle \in \vee^n(\mathbb{C}^{d_A} \otimes \ell^2(\mathcal{T}))$ is an n-qudit (symmetric) state with position information (where $|\Psi_0\rangle = |\Omega\rangle$). An orthonormal basis of $\mathcal{F}_A^{\mathcal{T}}$ is given by the union of a basis for each $\vee^n(\mathbb{C}^{d_A} \otimes \ell^2(\mathcal{T}))$, and an orthonormal basis for $\vee^n(\mathbb{C}^{d_A} \otimes \ell^2(\mathcal{T}))$ is given by all multisets[26] of cardinality n of elements from $\{|v_t\rangle : v \in \mathcal{V}, t \in \mathcal{T}\}$.

---

[26]Due to the restriction to a symmetric subspace, only sets with repetition (multisets) not sequences of basis elements of $\mathbb{C}^{d_A} \otimes \ell^2(\mathcal{T})$ are relevant.

Remark A.1. Throughout this paper we use Eq. (9) to split a wire into the tensor product of two wires, or to merge the tensor product of two wires into one. Here we explicitly provide the isomorphism used in Eq. (9).

Let $\{|x\rangle : x \in \mathcal{A}^{\vee n}\}$ and $\{|x\rangle : x \in \mathcal{B}^{\vee n}\}$ denote bases of $\vee^n \mathcal{H}_A$ and $\vee^n \mathcal{H}_B$, respectively, as defined in Eqs. (35) and (36). A basis of $\vee^n(\mathcal{H}_A \oplus \mathcal{H}_B)$ is given by $\{|x\rangle : x \in (\mathcal{A} \oplus \mathcal{B})^{\vee n}\}$ for

$$(\mathcal{A} \oplus \mathcal{B})^{\vee n} = \{(i_1, \ldots, i_{n_1}, j_1, \ldots, j_{n_2}) : n_1 + n_2 = n, i_k \in \mathcal{A}, j_k \in \mathcal{B},$$
$$k < \ell \implies i_k \leq i_\ell \text{ and } j_k \leq j_\ell\}.$$

We define the isomorphism between $\mathcal{F}(\mathcal{H}_A) \otimes \mathcal{F}(\mathcal{H}_B)$ and $\mathcal{F}(\mathcal{H}_A \oplus \mathcal{H}_B)$ by setting

$$|(i_1, \ldots, i_{n_1})\rangle \otimes |(j_1, \ldots, j_{n_2})\rangle \cong |(i_1, \ldots, i_{n_1}, j_1, \ldots, j_{n_2})\rangle.$$

B Infinite dimensional representations of quantum operators

B.1 The Choi-Jamiołkowski representation

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite dimensional Hilbert spaces, and let $\Phi : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ be a CPTP map, where $\mathcal{L}(\mathcal{H})$ is the set of linear operators on $\mathcal{H}$. The Choi-Jamiołkowski representation of Φ is given by the operator $R_\Phi \in \mathcal{L}(\mathcal{H}_{BA})$ — the *Choi operator* — defined as

$$R_\Phi = \sum_{i,j} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|. \tag{37}$$

One can think of R_Φ as capturing the image of a basis of $\mathcal{L}(\mathcal{H}_A)$, namely $\{|i\rangle\langle j|\}_{i,j}$. R_Φ is positive semi-definite and satisfies

$$\operatorname{tr}_B R_\Phi = I_A. \tag{38}$$

In fact, any positive semi-definite operator satisfying Eq. (38) is the Choi-Jamiołkowski representation of some CPTP map [57].

For infinite dimensional spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the Choi operator of a map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ can be unbounded. Instead, the Choi-Jamiołkowski representation is defined as the sesquilinear positive semi-definite form[27] R_Φ on

$$\mathcal{H}_B \times \mathcal{H}_A = \operatorname{span}\{\psi_B \otimes \psi_A : \psi_B \in \mathcal{H}_B, \psi_A \in \mathcal{H}_A\},$$

satisfying

$$R_\Phi(\psi_B \otimes \psi_A; \varphi_B \otimes \varphi_A) \coloneqq \langle\psi_B|\Phi(|\bar{\psi}_A\rangle\langle\bar{\varphi}_A|)|\varphi_B\rangle, \tag{39}$$

---

[27] $R(\cdot; \cdot)$ is a sesquilinear form if it is antilinear in the first argument and linear in the second.

where $|\bar{\psi}\rangle = \sum_{i=1}^{\infty} |i\rangle\overline{\langle i|\psi\rangle}$ for some fixed basis $\{|i\rangle\}_i$ of $\mathcal{H}_A$ [58].

If the domain of the sesquilinear form R_Φ is the whole of $\mathcal{H}_B \otimes \mathcal{H}_A$, then the corresponding operator is bounded[28] (as in the case of finite spaces), and can be recovered as the operator $\hat{R}_\Phi \in \mathfrak{B}(\mathcal{H}_{BA})$ satisfying

$$\langle\psi_B| \otimes \langle\psi_A|\hat{R}_\Phi|\varphi_B\rangle \otimes |\varphi_A\rangle = R_\Phi(\psi_B \otimes \psi_A; \varphi_B \otimes \varphi_A). \qquad (40)$$

Eq. (38) can be rewritten as

$$\sum_j R_\Phi(j_B \otimes \psi_A; j_B \otimes \varphi_A) = \langle\psi_A|\varphi_A\rangle \qquad (41)$$

for any basis $\{|j\rangle\}_j$ of $\mathcal{H}_B$, and holds in the infinite dimensional case as well. The converse also holds in the infinite dimensional case: any positive semi-definite sesquilinear form satisfying Eq. (41) uniquely defines a CPTP map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ [58].

If the map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ is completely positive, but not trace-preserving, then the Choi-Jamiołkowski representation of Φ is still defined as the sesquilinear positive semi-definite from R_Φ given by Eq. (39). But this form does not satisfy Eq. (41). Similarly, in the finite case the Choi operator of a non-trace-preserving map is still given by the positive operator from Eq. (37), but it does not satisfy Eq. (38).

B.2 The natural representation

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite dimensional Hilbert spaces. A linear map $\Phi : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ can be represented as a linear map $K_\Phi : \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}} \to \mathcal{H}_B \otimes \mathcal{H}_{\bar{B}}$ defined by

$$K_\Phi|i\rangle|j\rangle = \sum_{k,\ell} \alpha_{k\ell}|k\rangle|\ell\rangle : \Longleftrightarrow \Phi(|i\rangle\langle j|) = \sum_{k,\ell} \alpha_{k\ell}|k\rangle\langle\ell| \qquad (42)$$

for some bases $\{|i_A\rangle\}_i$ and $\{|k_B\rangle\}_k$ of $\mathcal{H}_A$ and $\mathcal{H}_B$. The transformation $\Phi \mapsto K_\Phi$ is a bijection, and is referred to as the *natural representation* [57].

In the infinite dimensional case, one may define the natural representation of a map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ in the same way. Note however that Eq. (42) is not well-defined on the entire space of linear maps $\Phi : \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_B)$. The inner product on $\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ is isomorphic to the Hilbert-Schmidt inner product[29] of operators on $\mathcal{H}_A$. The transformation $\Phi \mapsto K_\Phi$ given by Eq. (42) is thus a bijection between the set of maps $\mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ and the set of maps $\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}} \to \mathcal{H}_B \otimes \mathcal{H}_{\bar{B}}$, where $\mathfrak{S}(\mathcal{H})$ is the space of all operators on $\mathcal{H}$ with a bounded Hilbert-Schmidt norm. Since $\mathfrak{T}(\mathcal{H}) \subseteq \mathfrak{S}(\mathcal{H})$, this

---

[28]We denote the set of bounded operators on $\mathcal{H}$ by $\mathfrak{B}(\mathcal{H})$. $U \in \mathfrak{B}(\mathcal{H})$ if there exists some $c > 0$ such that for all $\psi \in \mathcal{H}$, $\|U\psi\|/\|\psi\| < c$.

[29]The Hilbert-Schmidt inner product is defined as $\langle A, B\rangle := \text{tr}(A^\dagger B)$.

representation is in particular well-defined for linear transformations between trace class operators, $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$.

The natural and Choi-Jamiołkowski representations are related as follows.

$$\begin{aligned}
\langle \psi_B | \langle \varphi_{\bar{B}} | K_\Phi | \psi_A \rangle | \varphi_{\bar{A}} \rangle &= \langle \psi_B | \Phi(|\psi_A\rangle\langle\bar{\varphi}_A|) | \bar{\varphi}_B \rangle \\
&= R_\Phi(\psi_B \otimes \bar{\psi}_A; \bar{\varphi}_B \otimes \varphi_A) ,
\end{aligned} \tag{43}$$

where, as previously, $|\bar{\psi}\rangle = \sum_{i=1}^\infty |i\rangle \overline{\langle i|\psi\rangle}$ for some fixed basis $\{|i\rangle\}_i$.

We now prove that the formula for a loop from Eq. (31) may be equivalently written using the natural representation, as noted in Remark 6.4.

Lemma B.1. *Let $\Phi : \mathfrak{T}(\mathcal{H}_{AC}) \to \mathfrak{T}(\mathcal{H}_{BC})$ be a linear map. Let $\Psi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ be another linear map such that for some bases $\{k_C\}_k$ and $\{\ell_C\}_\ell$ of $\mathcal{H}_C$,*

$$R_\Psi(\psi_B \otimes \psi_A; \varphi_B \otimes \varphi_A) =$$
$$\sum_{k,\ell} R_\Phi(\psi_B \otimes k_C \otimes \psi_A \otimes \bar{k}_C; \varphi_B \otimes \ell_C \otimes \varphi_A \otimes \bar{\ell}_C) .$$

Then

$$K_\Psi = \mathrm{tr}_{C\bar{C}}(K_\Phi) .$$

Proof. From Eq. (43) we have

$$R_\Psi(\psi_B \otimes \psi_A; \varphi_B \otimes \varphi_A) = \langle \psi_B | \langle \bar{\varphi}_{\bar{B}} | K_\Psi | \bar{\psi}_A \rangle | \varphi_{\bar{A}} \rangle .$$

And

$$\sum_{k,\ell} R_\Phi(\psi_B \otimes k_C \otimes \psi_A \otimes \bar{k}_C; \varphi_B \otimes \ell_C \otimes \varphi_A \otimes \bar{\ell}_C)$$
$$= \sum_{k,\ell} \langle \psi_B | \langle \bar{\varphi}_{\bar{B}} | \langle k_C | \langle \bar{\ell}_{\bar{C}} | K_\Phi | \bar{\psi}_A \rangle | \varphi_{\bar{A}} \rangle | k_C \rangle | \bar{\ell}_{\bar{C}} \rangle$$
$$= \langle \psi_B | \langle \bar{\varphi}_{\bar{B}} | \, \mathrm{tr}_{C\bar{C}}(K_\Phi) | \bar{\psi}_A \rangle | \varphi_{\bar{A}} \rangle . \qquad \square$$

C Finite causal boxes

In this work causal boxes are defined by a set of maps. As explained in Section 4.1, this allows systems to be included which produce an unbounded number of messages and are thus not well-defined as a single map on the entire set $\mathcal{T}$, but only on all subsets $\mathcal{T}^{\le t}$ for any $t \in \mathcal{T}$. In this section we define the subset of causal boxes that are defined on the entire set $\mathcal{T}$ and which are closed under composition. We call these *finite* causal boxes.

It is not sufficient to define finite causal boxes as those systems that are captured by a map $\Phi : \mathfrak{T}(\mathcal{F}_X^\mathcal{T}) \to \mathfrak{T}(\mathcal{F}_Y^\mathcal{T})$ since this set is not closed under composition. For example, consider a system with $\mathcal{T} = \mathbb{N}_0$, one input wire

and two output wires, which, for every qubit received in position $t \in \mathbb{N}_0$, outputs a qubit on each wire in position $t + 1$. Furthermore, this system outputs a qubit on each wire at position $t = 0$. The system can be described by a map $\Phi : \mathfrak{T}(\mathcal{F}_X^{\mathcal{T}}) \to \mathfrak{T}(\mathcal{F}_Y^{\mathcal{T}})$ which outputs two sequences of $n + 1$ qubits for every sequence of n qubits it receives at positions shifted by 1. If we now put a loop from one of the output wires to the input wire, we get a system with one output wire that produces a qubit at every position $t \in \mathbb{N}_0$. This new system outputs an infinite sequence of qubits, which is well-defined for every $t \in \mathcal{T}$, but not on the entire set $\mathcal{T}$.

The causality function (Definition 4.3) associated with each causal box guarantees that every point $t' \in \mathcal{T}$ may be reached from any other point $t \leq t'$ in a finite number of causal steps. To obtain closure under composition for finite causal boxes, it is necessary to limit causal boxes to a finite behavior, i.e., that all of $\mathcal{T}$ may be reached in a finite number of causal steps from any $t \in \mathcal{T}$. The set of points $\mathcal{T} \setminus \chi(\mathcal{T})$ are not needed to generate outputs, so one can think of causal boxes as terminating— at least, not reading further inputs— when a point in $\mathcal{T} \setminus \chi(\mathcal{T})$ has been reached.

Definition C.1 (Finite causality function). A causality function $\chi : \mathfrak{C}(\mathcal{T}) \to \mathfrak{C}(\mathcal{T})$ is a *finite causality function* if for every $t \in \mathcal{T}$ there exists an $n \in \mathbb{N}$ such that

$$t \notin \chi^n(\mathcal{T}) \,. \tag{44}$$

We can now define a finite causal box.

Definition C.2 (Finite causal box). A (d_X, d_Y)-*finite causal box* Φ is a system with input wire X and output wire Y of dimension d_X and d_Y, defined by a completely positive, trace-preserving (CPTP) map

$$\Phi : \mathfrak{T}\big(\mathcal{F}_X^{\mathcal{T}}\big) \to \mathfrak{T}\big(\mathcal{F}_Y^{\mathcal{T}}\big)$$

satisfying the following causality constraint: there must exist a function $\chi : \mathfrak{C}(\mathcal{T}) \to \mathfrak{C}(\mathcal{T})$ satisfying Definition C.1 such that for all $\mathcal{C} \in \mathfrak{C}(\mathcal{T})$,

$$\Phi^{\mathcal{C}} = \Phi^{\mathcal{C}} \circ \mathrm{tr}_{\mathcal{T} \setminus \chi(\mathcal{C})} \,, \tag{45}$$

where $\Phi^{\mathcal{C}} := \mathrm{tr}_{\mathcal{T} \setminus \mathcal{C}} \circ \Phi$.

Eq. (45) may be rewritten as

$$\mathrm{tr}_{T \setminus \mathcal{C}} \circ \Phi = \Phi^{\mathcal{C}} \circ \mathrm{tr}_{\mathcal{T} \setminus \chi(\mathcal{C})} \,, \tag{46}$$

which is an exact replica of Eq. (21) with $\mathcal{D} = \mathcal{T}$ and $\mathcal{C} \in \mathfrak{C}(\mathcal{T})$ instead of $\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})$. The Stinespring and Choi-Jamiołkowski representations developed in Section 5 for Eq. (21) also apply to Eq. (46). In particular, the map Φ of a finite causal box has a sequence representation (Definition 5.3). It then

follows that the proofs of closure for causal boxes from Section 6 are also proofs of closure for finite causal boxes.

The proof of composition order indepence from Section 6 and the pseudometric definition from Section 7 are valid for all causal boxes and thus in particular for finite causal boxes. Note that the causal boxes used to construct the distinguishers from Definition 7.1 are actually finite causal boxes.

D Causality for total orders

In Definition 4.3 a causality function is defined to guarantee that every point $t' \in \mathcal{T}$ can be reached from any point $t \leq t'$ in a finite number of causal steps. We show in this section that in the special case where $\mathcal{T}$ is totally ordered, this condition can be reduced to requiring that the system is not arbitrarily fast around any point.

More precisely, for a system Φ, let $\chi : \mathcal{T} \to \mathcal{T} \cup \{\bot\}$ be a monotone function such that the output up to position t can be computed from the input up to position $\chi(t) < t$, where $\chi(t) = \bot$ means that the output up to position t can be computed without any inputs.[30] As illustrated in Section 4.3, the condition $\chi(t) < t$ is not sufficient to guarantee that systems are closed under composition: an example was given in which an infinite number of causal steps were performed before reaching the point $t = 1$. We exclude such systems by additionally requiring that $\chi(t) < t$ must also hold in the limit as $t \to t_0$, i.e., $\inf_{t > t_0} \chi(t) < \inf_{t > t_0} t$ and $\sup_{t < t_0} \chi(t) < \sup_{t < t_0} t$.[31]

In the example discussed in Section 4.3, the sequence of output positions $\{0, 1/2, 3/4, 7/8, 15/16, \dots\}$ converges to 1, which is a point in $\mathcal{T} = \mathbb{Q}^+$. One could however give an example in which the sequence converges to a point which is not in $\mathcal{T}$, e.g., $\sqrt{2}$. Hence we require that the systems are not arbitrarily fast around all points t_0 in the completion of $\mathcal{T}$ (see Definition D.1 here below and [67] for the definition of a completion of an ordered set). The condition with the supremum is actually redundant, so we omit it from the following definition.

Definition D.1 (Causality function for totally ordered sets[32])**.** Let $\mathcal{T}$ be a totally ordered countable set. Let $\mathfrak{Z}(\mathcal{T}) \supseteq \mathcal{T}$ be its smallest completion such that for every subset $\mathcal{P} \subseteq \mathcal{T}$, $\inf \mathcal{P} \in \mathfrak{Z}(\mathcal{T})$ and $\sup \mathcal{P} \in \mathfrak{Z}(\mathcal{T})$.[33] If $\mathcal{T}$ does not have a minimum (maximum), we remove the infimum (supremum) from

---

[30]By defining $\hat{\chi} : \mathfrak{C}(\mathcal{T}) \to \mathfrak{C}(\mathcal{T})$ as $\hat{\chi}(\mathcal{C}) := \bigcup_{t \in \mathcal{C}} \mathcal{T}^{\leq \chi(t)}$ one gets a function that trivially satisfies all requirements of Definition 4.3 except Eq. (18).

[31]In general there might not be any distance measure on $\mathcal{T}$, so these criteria are defined using infimum and supremum instead of a limit.

[32]In the case where $\mathcal{T} \subseteq \mathbb{R}$, Definition D.1 is equivalent to requiring that for every $u \in \mathcal{T}$ there exists a $\delta_u > 0$ such that for all $t \leq u$, $t - \chi(t) > \delta_u$.

[33]This is called the Dedekind-MacNeille completion [67]. For example, $\mathfrak{Z}(\mathbb{N}) = \mathbb{N} \cup \{+\infty\}$ and $\mathfrak{Z}(\mathbb{Q}) = \mathbb{R} \cup \{-\infty, +\infty\}$.

$\mathfrak{Z}(\mathcal{T})$, i.e., for $p \in \{\inf \mathcal{T}, \sup \mathcal{T}\}$ we define

$$\overline{\mathfrak{Z}}(\mathcal{T}) \coloneqq \begin{cases} \mathfrak{Z}(\mathcal{T}) \setminus \{p\} & \text{if } p \notin \mathcal{T}, \\ \mathfrak{Z}(\mathcal{T}) & \text{if } p \in \mathcal{T}. \end{cases}$$

Finally, let $\perp$ be a point defined such that $\perp < t_0$ for all $t_0 \in \overline{\mathfrak{Z}}(\mathcal{T})$. A montone function $\chi : \mathcal{T} \to \mathcal{T} \cup \{\perp\}$ is a *causality function for totally ordered sets* if

$$\forall t \in \mathcal{T}, \quad \chi(t) < t,$$
$$\forall t_0 \in \overline{\mathfrak{Z}}(\mathcal{T}), \quad \inf_{t > t_0} \chi(t) < \inf_{t > t_0} t.$$

With the causality function defined as $\chi : \mathcal{T} \to \mathcal{T} \cup \{\perp\}$ instead of $\chi : \mathfrak{C}(\mathcal{T}) \to \mathfrak{C}(\mathcal{T})$, Eq. (18) from Definition 4.3 can be rewritten as

$$\forall t, t' \in \mathcal{T}, \exists n \in \mathbb{N}, \quad \chi^n(t') < t. \tag{47}$$

We prove in Lemma D.2 that Definition D.1 implies that Eq. (47) is satisfied.

Lemma D.2. *Let $\chi : \mathcal{T} \to \mathcal{T} \cup \{\perp\}$ be a function satisfying Definition D.1. Then for all $t, t' \in \mathcal{T}$ there exists an $n \in \mathbb{N}$ such that $\chi^n(t') \le t$.*

Proof. By contradiction, assume that this is not the case for some $t, t' \in \mathcal{T}$ and consider the set $\mathcal{P} = \{\chi^n(t')\}_{n \in \mathbb{N}}$. By the construction of $\overline{\mathfrak{Z}}(\mathcal{T})$, the completion of $\mathcal{T}$, $\inf \mathcal{P} \in \overline{\mathfrak{Z}}(\mathcal{T})$. Let $t_0 \coloneqq \inf \mathcal{P}$. Because χ is monotone and $\chi(p) < p$, we must have that for all $p > t_0$, $\chi(p) > t_0$. On the other hand, the condition $\inf_{p > t_0} \chi(p) < \inf_{p > t_0} t$ implies that there exists $p > t_0$ such that $\chi(p) \le t_0$. $\square$

We now strengthen the causality definition from Definition D.1 so that it captures sufficient conditions to define finite causal boxes on totally ordered sets. As in Appendix C we have to exclude unbounded behavior as t gets larger. We achieve this by requiring that the systems may not be arbitrarily fast around the supremum of $\mathcal{T}$.

Definition D.3 (Finite causality function for totally ordered sets)**.** Let $\chi : \mathcal{T} \to \mathcal{T} \cup \{\perp\}$ be a function satisfying Definition D.1. We say that it is a *finite causality function for totally ordered sets* if additionally,

$$\sup_{t < t_0} \chi(t) < \sup_{t < t_0} t$$

for $t_0 \coloneqq \sup \mathcal{T}$.

With causality functions defined as $\chi : \mathcal{T} \to \mathcal{T} \cup \{\perp\}$, the condition for finite causal boxes from Eq. (44) in Definition C.1 becomes

$$\forall t \in \mathcal{T}, \exists n \in \mathbb{N}, \forall t' \in \mathcal{T}, \quad \chi^n(t') < t. \tag{48}$$

We now prove that this is satisfied by Definition D.3.

Lemma D.4. *Let $\chi : \mathcal{T} \to \mathcal{T} \cup \{\perp\}$ be a function satisfying Definition D.3. Then for every $t \in \mathcal{T}$ there exists an $n \in \mathbb{N}$ such that for all $t' \in \mathcal{T}$, $\chi^n(t') < t$.*

Proof. If $\mathcal{T}$ has a maximum $t_{\max}$, the lemma follows trivially, because by Lemma D.2 for all t there exists an n such that $\chi^n(t_{\max}) < t$, and for all t', $\chi^n(t') \leq \chi^n(t_{\max})$. In the case where $\mathcal{T}$ does not have a maximum, let $t_{\sup} = \sup \mathcal{T}$ and let $p = \sup_{t < t_{\sup}} \chi(t)$. From Definition D.3 we have $p < t_{\sup}$, so there must exist $t_0 \in \mathcal{T}$ such that $t_0 \geq p$. Furthermore, for all $t \geq t_0$, $\chi(t) \leq p \leq t_0$. Finally, by Lemma D.2 for all t there exists an n such that $\chi^n(t_0) < t$, hence for all $t' \in \mathcal{T}$, $\chi^{n+1}(t') < t$. $\qquad\square$

E   Loops

In Section 6.2 we proposed an alternative, constructive defintion for a loop (Eq. (30)) as the limit of a sequence of maps. Each element in this sequence loops a bit more of the output back to the input, and the limit defines the complete loop. For a sequence of operators, the limit is defined as follows.

Definition E.1 (Operator convergence)**.** Let $\{U_i : \mathcal{H}_X \to \mathcal{H}_Y\}_{i=1}^{\infty}$ be a sequence of linear operators. We say that U_i converges to U (in the strong operator topology) if and only if

$$\forall \psi \in \mathcal{H}_X, \forall \varepsilon > 0, \exists i_0, \forall i \geq i_0, \quad \|U_i \psi - U \psi\| < \varepsilon \,,$$

where $\| \cdot \|$ is the 2-norm.

Let $\Phi^{\mathcal{C}_1} : \mathfrak{T}(\mathcal{F}_{AB}^{\mathcal{C}_2}) \to \mathfrak{T}(\mathcal{F}_{CD}^{\mathcal{C}_1})$ be a map with sequence representation

$$U_{\Phi}^{\mathcal{C}_1} = \left(\prod_{i=1}^{n-1} I_{CD}^{\mathcal{C}_{i+1}} \otimes V_i \otimes I_{AB}^{\mathcal{C}_2 \backslash \mathcal{C}_{i+1}} \right) \left(U_{\Phi}^{\mathcal{C}_n} \otimes I_{AB}^{\mathcal{C}_2 \backslash \mathcal{C}_{n+1}} \right) , \qquad (49)$$

which is illustrated in Figure 13a. We define the linear operator

$$U_{\Psi_n}^{\mathcal{C}_1} := \left(\langle \Omega |_C^{\mathcal{C}_{n+1}} \otimes I_D^{\mathcal{C}_1} \otimes I_{Q_1} \otimes I_C^{\mathcal{T}_1} \right) \left(\prod_{i=1}^{n-1} I_D^{\mathcal{C}_{i+1}} \otimes V_i \otimes I_A^{\mathcal{C}_2 \backslash \mathcal{C}_{i+1}} \right)$$
$$\left(U_{\Phi}^{\mathcal{C}_n} \otimes I_A^{\mathcal{C}_2 \backslash \mathcal{C}_{n+1}} \right) \left(|\Omega\rangle_B^{\mathcal{C}_{n+1}} \otimes I_A^{\mathcal{C}_2} \right) , \quad (50)$$

which is obtained from Eq. (49) by looping the output on $\mathcal{F}_C^{\mathcal{T}_i}$ to the input on $\mathcal{F}_B^{\mathcal{T}_i}$ for $i \leq n$, inputting the vacuum state $|\Omega\rangle$ on the sub-wire $\mathcal{F}_B^{\mathcal{C}_{n+1}}$ and projecting the sub-wire $\mathcal{F}_C^{\mathcal{C}_{n+1}}$ on the vacuum state—the input $|\Omega\rangle_B^{\mathcal{C}_{n+1}}$ and projection on $|\Omega\rangle_C^{\mathcal{C}_{n+1}}$ ensure that all operators $\{U_{\Psi_n}^{\mathcal{C}_1}\}_{n=1}^{\infty}$ have the same input and output Hilbert spaces. This is depicted in Figure 16, which is a reproduction of Figure 13b along with the extra projections on $|\Omega\rangle_C^{\mathcal{C}_{n+1}}$ and
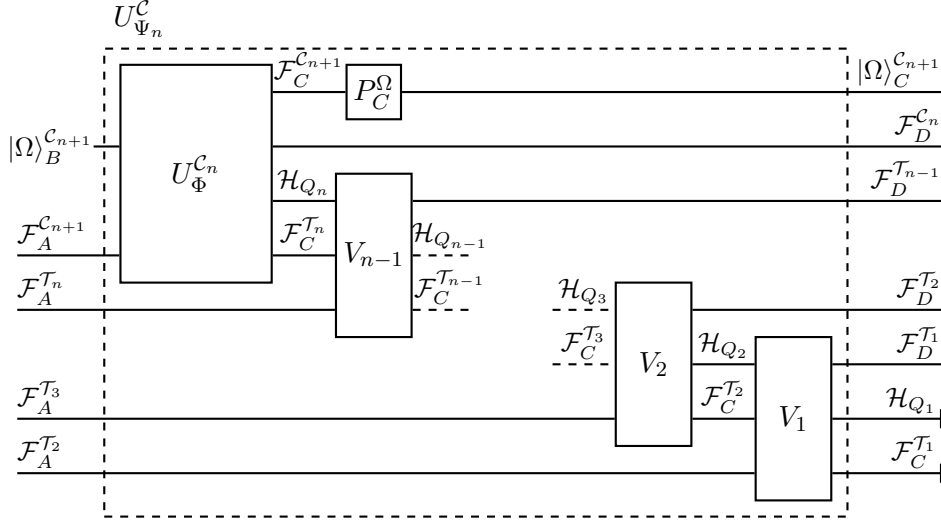
Figure 16 – The outputs on C in positions $\mathcal{T}_i$ for $i \leq n$ are looped back to the inputs on B. The outputs on C for $i > n$ are projected on the vacuum state—P_C^Ω denotes this projector—and a vacuum state is input on B in the corresponding positions.

vacuum inputs $|\Omega\rangle_B^{\mathcal{C}_{n+1}}$, that had been omitted for simplicity. Let $U_\Psi^{\mathcal{C}_1}$ be the limit operator as more of the wire C is looped back to the input B, i.e.,

$$U_\Psi^{\mathcal{C}_1} := \lim_{n \to \infty} U_{\Psi_n}^{\mathcal{C}_1}.$$

And define the map $\Psi^{\mathcal{C}_1} : \mathfrak{T}(\mathcal{F}_A^{\mathcal{C}_2}) \to \mathfrak{T}(\mathcal{F}_D^{\mathcal{C}_1})$ by

$$\Psi^{\mathcal{C}_1}(\rho) := \mathrm{tr}_{Q_1 C_1}\left(U_\Psi^{\mathcal{C}_1} \rho \left(U_\Psi^{\mathcal{C}_1}\right)^\dagger\right), \tag{51}$$

where $\mathrm{tr}_{Q_1 C_1}$ traces out the ancilla register Q_1 as well as the sub-wire with Hilbert space $\mathcal{F}_C^{\mathcal{T}_1}$. The main proposition that we prove in this section is that the map $\Psi^{\mathcal{C}_1}$ defined above is the map one obtains by applying the definition of a loop (Definition 6.3) to $\Phi^{\mathcal{C}_1}$.

Proposition E.2. *The Choi-Jamiołkowski representation of the map $\Psi^{\mathcal{C}_1}$ from Eq. (51) is given by Eq. (31). Furthermore, the terms in Eq. (31) are absolutely convergent, and if $\Phi^{\mathcal{C}_1}$ is CPTP, then so is $\Psi^{\mathcal{C}_1}$.*

We prove this proposition with the help of two lemmas. The first simply states that the operators $U_{\Psi_n}^{\mathcal{C}_1}$ converge.

Lemma E.3. *Let the operators $U_{\Psi_n}^{\mathcal{C}_1}$ be defined as above. Then the limit operator $U_\Psi^{\mathcal{C}_1} = \lim_{n \to \infty} U_{\Psi_n}^{\mathcal{C}_1}$ exists. Furthermore, if $U_\Phi^{\mathcal{C}_1}$ is an isometry, then so is $U_\Psi^{\mathcal{C}_1}$.*

Proof. We prove the case in which $U_\Phi^{\mathcal{C}_1}$ is an isometry. The case of a linear operator $U_\Phi^{\mathcal{C}_1}$ that is not an isometry follows by writing it as an isometry followed by a projection.

Let $\{b_i^j\}_i$ be a basis of $\mathcal{F}_C^{\mathcal{T}_j}$ and let Ω^n be shorthand for the vacuum state $|\Omega\rangle_C^{\mathcal{C}_n}$. We build a basis for $\mathcal{F}_C^{\mathcal{C}_1}$ consisting of states that have a vacuum as prefix, i.e., all states of the form $c_i = \Omega^n \otimes \bigotimes_{j=1}^{n-1} b_{i_j}^j$, where $b_{i_j}^j$ is a basis state of $\mathcal{F}_C^{\mathcal{T}_j}$ and $n \in \mathbb{N}$. Any state $\varphi \in \mathcal{F}_C^{\mathcal{C}_1}$ can thus be written as $\varphi = \sum_i \alpha_i c_i$ for some coefficients $\alpha_i \in \mathbb{C}$. For a normalized state $\varphi \in \mathcal{F}_C^{\mathcal{C}_1}$ and any $\varepsilon > 0$ one can always find a finite set of basis states indexed by $i \in \mathcal{S}$ such that $\sum_{i \in \mathcal{S}} |\alpha_i|^2 \geq 1 - \varepsilon$. Let n_0 be such that for all $i \in \mathcal{S}$, $c_i = \Omega^{n_i} \otimes c_i'$ with $n_i \leq n_0$. This means that $\varphi = \Omega^{n_0} \otimes \varphi_{n_0-1} + \varphi'$, where $\varphi_{n_0-1} \in \bigotimes_{j=1}^{n_0-1} \mathcal{F}_C^{\mathcal{T}_j}$ and $\|\varphi'\|^2 \leq \varepsilon$.

We fix $\varepsilon > 0$ and $\psi \in \mathcal{F}_A^{\mathcal{C}_2}$. To prove that the operators $U_{\Psi_n}^{\mathcal{C}_1}$ converge, we will show that there exists an n_0 such that for any $n, m \geq n_0$,

$$\left\| U_{\Psi_n}^{\mathcal{C}_1} \psi - U_{\Psi_m}^{\mathcal{C}_1} \psi \right\| \leq \sqrt{\varepsilon}.$$

Let $\varphi := U_\Phi^{\mathcal{C}_1}(\psi_A \otimes |\Omega\rangle_B^{\mathcal{C}_2}) \in \mathcal{F}_{CD}^{\mathcal{C}_1}$ and let $n_0 \in \mathbb{N}$ be such that $\varphi = \Omega^{n_0} \otimes \varphi_{n_0-1} + \varphi'$, where $\varphi_{n_0-1} \in \bigotimes_{j=1}^{n_0-1} \mathcal{F}_C^{\mathcal{T}_j} \otimes \mathcal{F}_D^{\mathcal{C}_1}$ and $\|\varphi'\|^2 \leq \varepsilon$. Let $n \geq m \geq n_0$. The decompositions of $U_{\Psi_n}^{\mathcal{C}_1}$ and $U_{\Psi_m}^{\mathcal{C}_1}$ into sequences of operators as in Figure 16 both start with $U_\Phi^{\mathcal{C}_n}$ and finish with the sequence V_{m-1} to V_1. They differ only in their behavior on the positions $\mathcal{C}_m \setminus \mathcal{C}_n$. Here, $U_{\Psi_n}^{\mathcal{C}_1}$ consists in the sequence V_{n-1} to V_m that are applied to the values on the A and C wires as well as the internal memory with Hilbert spaces $\mathcal{H}_{Q_i}$, whereas the sequence decomposition of the operator $U_{\Psi_m}^{\mathcal{C}_1}$ additionally contains projectors on the vacuum state and the operators V_i, $i \in \{m, \ldots, n-1\}$, are only applied to input states that have a vacuum on the B wire. Because these are linear operators we have $U_{\Psi_m}^{\mathcal{C}_1} \psi = \Omega^{m+1} \otimes \varphi_m$ and $U_{\Psi_n}^{\mathcal{C}_1} \psi = \Omega^{m+1} \otimes \varphi_m + \varphi'$. And because $m \geq n_0$, φ_m must have weight $\|\varphi_m\|^2 \geq 1 - \varepsilon$, hence $\|\varphi'\|^2 \leq \varepsilon$. Putting this together we get

$$\left\| U_{\Psi_n}^{\mathcal{C}_1} \psi - U_{\Psi_m}^{\mathcal{C}_1} \psi \right\| = \left\| \Omega^{m+1} \otimes \varphi_m + \varphi' - \Omega^{m+1} \otimes \varphi_m \right\|$$
$$= \left\| \varphi' \right\| \leq \sqrt{\varepsilon}.$$

It is immediate that the limit operator $U_\Psi^{\mathcal{C}_1}$ is an isometry, since for any ψ with $\|\psi\| = 1$ and any ε there exists n_0 such that for any $n \geq n_0$, $\|U_{\Psi_n}^{\mathcal{C}_1} \psi\| \geq 1 - \varepsilon$. $\qquad \square$

We now need to relate the maps $\Psi_n^{\mathcal{C}_1}$ which have the Stinespring representations $U_{\Psi_n}^{\mathcal{C}_1}$ to their Choi-Jamiołkowski representations. In the following lemma we do this first for a map $\Phi : \mathfrak{T}(\mathcal{H}_{AB}) \to \mathfrak{T}(\mathcal{H}_{CD})$ for which the output on C does not depend on the input on B. This special case was

illustrated in Figure 12, where U_Φ is depicted on the left and the resulting system after connecting C to B is drawn on the right.

Lemma E.4. *Let $\Phi : \mathfrak{T}(\mathcal{H}_{AB}) \to \mathfrak{T}(\mathcal{H}_{CD})$ be a CP map with a Stinespring representation given by*

$$U_\Phi = (I_C \otimes V)(U_\Psi \otimes I_B),$$

where $U_\Psi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_{CQ})$ and $V : \mathfrak{T}(\mathcal{H}_{QB}) \to \mathfrak{T}(\mathcal{H}_{DR})$. Let $\{|k_C\rangle\}_k$ and $\{|\ell_C\rangle\}_\ell$ be any orthonormal bases of $\mathcal{H}_C$ and let $\{|k_B\rangle\}_k$ and $\{|\ell_B\rangle\}_\ell$ denote the corresponding bases of $\mathcal{H}_B$, i.e., for all k and ℓ, $|k_C\rangle \cong |k_B\rangle$ and $|\ell_C\rangle \cong |\ell_B\rangle$. The map $\Phi' : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_D)$ obtained by plugging C into B has Choi-Jamiołkowski representation

$$R_{\Phi'}(\psi_D \otimes \psi_A; \varphi_D \otimes \varphi_A) =$$
$$\sum_{k,\ell} R_\Phi(k_C \otimes \psi_D \otimes \psi_A \otimes \bar{k}_B; \ell_C \otimes \varphi_D \otimes \varphi_A \otimes \bar{\ell}_B), \quad (52)$$

where $|\bar{k}_B\rangle = \sum_{i=1}^\infty |i\rangle \overline{\langle i|k\rangle}$ for the basis $\{|i_B\rangle\}_i$ of $\mathcal{H}_B$ used in the Choi-Jamiołkowski representation of Φ. Furthermore, the terms in this sum are absolutely convergent.

Proof. Define $\bar{\Psi} : \mathfrak{T}(\mathcal{H}_{AB}) \to \mathfrak{T}(\mathcal{H}_{CQB})$ and $\bar{\Gamma} : \mathfrak{T}(\mathcal{H}_{CQB}) \to \mathfrak{T}(\mathcal{H}_{CD})$ as

$$\bar{\Psi}(\rho) = (U_\Psi \otimes I_B)\rho\left(U_\Psi^\dagger \otimes I_B\right),$$
$$\bar{\Gamma}(\rho) = \mathrm{tr}_R\left[(I_C \otimes V)\rho\left(I_C \otimes V^\dagger\right)\right],$$

and $\bar{\Psi}' : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_{CQ})$ and $\bar{\Gamma}' : \mathfrak{T}(\mathcal{H}_{QB}) \to \mathfrak{T}(\mathcal{H}_D)$ as

$$\bar{\Psi}'(\rho) = U_\Psi \rho U_\Psi^\dagger,$$
$$\bar{\Gamma}'(\rho) = \mathrm{tr}_R\left(V \rho V^\dagger\right).$$

Then $\Phi = \bar{\Gamma} \circ \bar{\Psi}$ and $\Phi' = \bar{\Gamma}' \circ \bar{\Psi}'$. Writing up the Choi-Jamiołkowski representations of these two maps we get for any basis $\{|i_Q\rangle\}_i$ of Q,

$$R_\Phi(k_C \otimes \psi_D \otimes \psi_A \otimes \bar{k}_B; \ell_C \otimes \varphi_D \otimes \varphi_A \otimes \bar{\ell}_B)$$
$$= \langle k_C, \psi_D | \bar{\Gamma}\left(\bar{\Psi}\left(|\bar{\psi}_A, k_B\rangle\langle\bar{\varphi}_A, \ell_B|\right)\right)|\ell_C, \varphi_D\rangle$$
$$= \sum_{e,h,i,j,m,n} \langle k_C, \psi_D | \bar{\Gamma}(|e_C, i_Q, m_B\rangle\langle h_C, j_Q, n_B|)|\ell_C, \varphi_D\rangle$$
$$\langle e_C, i_Q, m_B | \bar{\Psi}\left(|\bar{\psi}_A, k_B\rangle\langle\bar{\varphi}_A, \ell_B|\right)|h_C, j_Q, n_B\rangle$$
$$= \sum_{i,j} \langle k_C, \psi_D | \bar{\Gamma}(|k_C, i_Q, k_B\rangle\langle\ell_C, j_Q, \ell_B|)|\ell_C, \varphi_D\rangle$$
$$\langle k_C, i_Q, k_B | \bar{\Psi}\left(|\bar{\psi}_A, k_B\rangle\langle\bar{\varphi}_A, \ell_B|\right)|\ell_C, j_Q, k_B\rangle$$
$$= \sum_{i,j} \langle \psi_D | \bar{\Gamma}'(|i_Q, k_B\rangle\langle j_Q, \ell_B|)|\varphi_D\rangle\langle k_C, i_Q | \bar{\Psi}'\left(|\bar{\psi}_A\rangle\langle\bar{\varphi}_A|\right)|\ell_C, j_Q\rangle,$$

where we have used that $\bar{\bar{\Psi}}$ and $\bar{\Gamma}$ perform identity on B and C, respectively. And for Φ' we obtain.

$$
\begin{aligned}
R_{\Phi'}&(\psi_D \otimes \psi_A; \varphi_D \otimes \varphi_A) \\
&= \langle \psi_D | \bar{\Gamma}'\big(\bar{\Psi}'\big(|\bar{\psi}_A\rangle\langle\bar{\varphi}_A|\big)\big)|\varphi_D\rangle \\
&= \sum_{i,j,k,\ell} \langle \psi_D | \bar{\Gamma}'(|i_Q, k_B\rangle\langle j_Q, \ell_B|)|\varphi_D\rangle \langle k_C, i_Q | \bar{\Psi}'\big(|\bar{\psi}_A\rangle\langle\bar{\varphi}_A|\big)|\ell_C, j_Q\rangle \\
&= \sum_{k,\ell} R_\Phi(k_C \otimes \psi_D \otimes \psi_A \otimes \bar{k}_B; \ell_C \otimes \varphi_D \otimes \varphi_A \otimes \bar{\ell}_B).
\end{aligned}
$$

Note that this holds for any orthonormal bases $\{|k_C\rangle\}_k$ and $\{|\ell_C\rangle\}_\ell$ of $\mathcal{H}_C$. Hence the order in the sum is irrelevant and the terms are absolutely convergent. $\qquad\square$

With these two lemmas, we can now prove Proposition E.2. We first use Lemma E.4 to show that the Choi-Jamiołkowski representation of the system after the loop, namely $\Psi^{\mathcal{C}}$, is given by Eq. (31). Since in Lemma E.3 we prove that the limit operator $U_\Psi^{\mathcal{C}}$ is an isometry, then $\Psi^{\mathcal{C}}$ must be CPTP.

Proof of Proposition E.2. Let $R_\Phi^{\mathcal{C}_1}(\cdot; \cdot)$ be the Choi-Jamiołkowski representation of $\Phi^{\mathcal{C}_1}$. We derive the Choi-Jamiołkowski representation of $\Psi_n^{\mathcal{C}_1}$ by repeating n times the proof from Lemma E.4, i.e., connecting $\mathcal{F}_C^{\mathcal{T}_1}$ to $\mathcal{F}_B^{\mathcal{T}_1}$, $\mathcal{F}_C^{\mathcal{T}_2}$ to $\mathcal{F}_B^{\mathcal{T}_2}$, etc. This yields

$$
\begin{aligned}
R_{\Psi_n}^{\mathcal{C}_1}&(\psi_D \otimes \psi_A; \varphi_D \otimes \varphi_A) = \\
&= \sum_{k,\ell} R_\Phi^{\mathcal{C}_1}\Big(\Omega_C^{\mathcal{C}_{n+1}} \otimes k_C \otimes \psi_D \otimes \psi_A \otimes \bar{\Omega}_B^{\mathcal{C}_{n+1}} \otimes \bar{k}_B; \\
&\qquad\qquad\qquad \Omega_C^{\mathcal{C}_{n+1}} \otimes \ell_C \otimes \varphi_D \otimes \varphi_A \otimes \bar{\Omega}_B^{\mathcal{C}_{n+1}} \otimes \bar{\ell}_B\Big),
\end{aligned}
$$

where $\{k_C\}_k$ and $\{\ell_C\}_\ell$ are bases of $\bigotimes_{i=1}^n \mathcal{F}_C^{\mathcal{T}_i} = \mathcal{F}_C^{\mathcal{C}_1 \backslash \mathcal{C}_{n+1}}$. As in Lemma E.4, the order of the summation is not relevant and so the terms are absolutely convergent. Using the fact that $\bigcup_{i=1}^\infty \mathcal{T}_i = \mathcal{C}_1$, we immediately get the limiting case,

$$
\begin{aligned}
R_\Psi^{\mathcal{C}_1}&(\psi_D \otimes \psi_A; \varphi_D \otimes \varphi_A) \\
&= \lim_{n\to\infty} R_{\Psi_n}^{\mathcal{C}_1}(\psi_D \otimes \psi_A; \varphi_D \otimes \varphi_A) \\
&= \sum_{k,\ell} R_\Phi^{\mathcal{C}_1}\big(k_C \otimes \psi_D \otimes \psi_A \otimes \bar{k}_B; \ell_C \otimes \varphi_D \otimes \varphi_A \otimes \bar{\ell}_B\big),
\end{aligned}
$$

where $\{k_C\}_k$ and $\{\ell_C\}_k$ are bases of $\mathcal{F}_C^{\mathcal{C}_1}$, and as above, the terms are absolutely convergent. Finally, from Lemma E.3 we know that $U_\Psi^{\mathcal{C}_1}$ is an isometry if $U_\Phi^{\mathcal{C}_1}$ is an isometry, hence $\Psi^{\mathcal{C}_1}$ is CPTP if $\Phi^{\mathcal{C}_1}$ is CPTP. $\qquad\square$

F   Alternative distinguishers

In this section we consider two changes that could be made to the notion of distinguishers and the corresponding pseudo-metric introduced in Section 7. In Appendix F.1 we define distinguishers as subnormalized causal boxes: instead of a normalized system which outputs either 0 or 1, we consider the subnormalized system resulting from conditioning the output on 0. This results in a simplification of the pseudo-metric definition. The second change discussed in Appendix F.2 consists in removing the constraint that a distinguisher $\mathcal{D}$ terminates at a fixed point $t_{\mathcal{D}}$. Instead we consider a possibly infinite sequence of bounded cuts $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots$, and define the distinguisher's output to be the limit value of the output on $\mathcal{C}_i$ as $i \to \infty$. We then prove that the resulting pseudo-metric is equivalent to the one from Section 7.

F.1   Subnormalized distinguishers

Let $\hat{\mathfrak{D}} = \{\hat{\mathcal{D}}\}$ be a set of distinguishers. Since by construction we define the output value of a distinguisher $\hat{\mathcal{D}}$ to be computed at some point $t_{\hat{\mathcal{D}}}$, the behavior of $\hat{\mathcal{D}}$ on all points $p \not\leq t_{\hat{\mathcal{D}}}$ is irrelevant, and one can assume that $\hat{\mathcal{D}}$ is entirely described by the map $\hat{\mathcal{D}}^{\leq t_{\hat{\mathcal{D}}}}$. For any $\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})$ one can take

$$\hat{\mathcal{D}}^{\mathcal{C}}(\rho) \coloneqq \mathrm{tr}_{\mathcal{T} \setminus \mathcal{C}} \left(\hat{\mathcal{D}}^{\leq t_{\hat{\mathcal{D}}}}(\rho) \otimes |\Omega\rangle\langle\Omega|^{\mathcal{T} \setminus \mathcal{T}^{\leq t_{\hat{\mathcal{D}}}}} \right). \tag{53}$$

Given such a set $\hat{\mathfrak{D}} = \{\hat{\mathcal{D}}_i\}_i$, we define a set of subnormalized distinguishers $\mathfrak{D} = \{\mathcal{D}_i\}_i$ by projecting the output of the distinguisher $\hat{\mathcal{D}}_i$ on the vacuum state $|\Omega\rangle$. Like for $\hat{\mathcal{D}}_i$, each $\mathcal{D}_i$ is entirely described by a subnormalized map $\mathcal{D}_i^{\leq t_{\mathcal{D}}}$, where $t_{\mathcal{D}} = t_{\hat{\mathcal{D}}}$. Using the construction from Eq. (53), any subnormalized map $\mathcal{D}^{\leq t_{\mathcal{D}}} = \mathrm{id}^{\leq t_{\mathcal{D}}} \otimes (\mathcal{D}')^{\leq t_{\mathcal{D}}}$ such that $(\mathcal{D}')^{\leq t_{\mathcal{D}}}$ satisfies causality is a valid subnormalized distinguisher.

Connecting a subnormalized (m, n)-distinguisher $\mathcal{D}$ to an (m, n)-causal boxes Φ results in a system $\mathcal{D}\Phi = \{\mathcal{D}^{\mathcal{C}}\Phi^{\mathcal{C}}\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ that has no output or input wires. It is thus a set of numbers, where $\mathcal{D}^{\leq t_{\mathcal{D}}}\Phi^{\leq t_{\mathcal{D}}} \in [0, 1]$ corresponds to the probability of $\hat{\mathcal{D}}$ outputting a vacuum state, i.e.,

$$\mathcal{D}^{\leq t_{\mathcal{D}}}\Phi^{\leq t_{\mathcal{D}}} = \Pr\left[\hat{\mathcal{D}}[\Phi] = 0\right].$$

Using the natural representation introduced in Remark 6.4, this probability may be written as a trace, namely

$$\mathcal{D}^{\leq t_{\mathcal{D}}}\Phi^{\leq t_{\mathcal{D}}} = \mathrm{tr}\left(K_{\mathcal{D}\Phi}^{\leq t_{\mathcal{D}}}\right),$$

where $K_{\mathcal{D}\Phi}^{\leq t_{\mathcal{D}}}$ is the natural representation of $\mathrm{SWAP}_{\mathcal{D}\Phi} \circ \left(\mathcal{D}^{\leq t_{\mathcal{D}}} \otimes \Phi^{\leq t_{\mathcal{D}}}\right)$, where $\mathrm{SWAP}_{\mathcal{D}\Phi}$ permutes the output wires so that they are aligned with the corresponding input wires and get connected by the trace operator tr.

The statistical distance between binary random variables may be written in terms of one of the outcomes:

$$\frac{1}{2} \sum_{x \in \{0,1\}} \left| \Pr\left[\hat{\mathcal{D}}[\Phi] = x \right] - \Pr\left[\hat{\mathcal{D}}[\Psi] = x \right] \right|$$

$$= \left| \Pr\left[\hat{\mathcal{D}}[\Phi] = 0 \right] - \Pr\left[\hat{\mathcal{D}}[\Psi] = 0 \right] \right|.$$

Hence the distinguisher pseudo-metric from Definition 7.2 may equivalently be written as

$$d^{\mathfrak{D}}(\Phi, \Psi) = \sup_{\mathcal{D} \in \mathfrak{D}} \left| \text{tr}\left(K_{\mathcal{D}\Phi}^{\leq t_{\mathcal{D}}} \right) - \text{tr}\left(K_{\mathcal{D}\Psi}^{\leq t_{\mathcal{D}}} \right) \right|,$$

where $t_{\mathcal{D}}$ is the position at which the output value of $\mathcal{D}$ is computed.

Putting this together we get an alternative definition of the distance between causal boxes which is equivalent to Definition 7.2.

Definition F.1 (Distance). Let $\mathfrak{D} = \{\mathcal{D}\}$ be a set of subnormalized distinguishers as described above. Let Φ and Ψ be two (m, n)-causal boxes. The distance between Φ and Ψ with respect to $\mathfrak{D}$ is given by

$$d^{\mathfrak{D}}(\Phi, \Psi) := \sup_{\mathcal{D} \in \mathfrak{D}} \left| \text{tr}\left(K_{\mathcal{D}\Phi}^{\leq t_{\mathcal{D}}} \right) - \text{tr}\left(K_{\mathcal{D}\Psi}^{\leq t_{\mathcal{D}}} \right) \right|.$$

F.2 The distinguishing limit

In Section 7 we only consider the output of a distinguisher $\mathcal{D}$ up to some point $t_{\mathcal{D}} \in \mathcal{T}$. In this section we consider an alternative definition of a distinguisher that is not constrained in such a way. Instead, we consider the output on a totally ordered (possibly infinite) sequence of bounded cuts $\{\mathcal{C}_i\}_{i \in \mathcal{I}}$. The output of the distinguisher is then taken to be the limit value as the sequence progresses. For example, if the set $\mathcal{T}$ is totally ordered, then one can choose $\mathcal{I} = \mathcal{T}$ and $\mathcal{C}_i = \mathcal{T}^{\leq i}$.

Definition F.2 (Limit distinguisher). Let $\mathcal{I}$ be a totally ordered set with $\infty := \sup \mathcal{I}$ and let $\{\mathcal{C}_i\}_{i \in \mathcal{I}}$ be a sequence of bounded cuts such that $i \leq j \implies \mathcal{C}_i \subseteq \mathcal{C}_j$. A (m, n)-*limit distinguisher* $\mathcal{D} = \{\text{id}^{\mathcal{C}} \otimes \hat{\mathcal{D}}^{\mathcal{C}}\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ consists of a $(\hat{n}, \hat{m})$-causal box $\hat{\mathcal{D}} = \{\hat{\mathcal{D}}^{\mathcal{C}}\}_{\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})}$ with $m + 1 - \hat{m} = n - \hat{n}$, a sequence of bounded cuts $\{\mathcal{C}_i\}_{i \in \mathcal{I}}$ as described above, and a specification of how the distinguisher is connected to an (m, n)-dimensional system — i.e., which input and output sub-wires are connected to $\hat{\mathcal{D}}$ and which are directly connected by a loop. For an (m, n)-distinguisher $\mathcal{D}$ and an (m, n)-causal box Φ, let $\mathcal{D}\Phi$ denote the causal box with no input wire and a 1-dimensional output wire resulting from connecting the systems as specified. We define $\mathcal{D}^{\mathcal{C}}[\Phi^{\mathcal{C}}]$ to be the binary random variable on $\{0, 1\}$ obtained by projecting the output of

$\mathcal{D}\Phi$ within $\mathcal{C}$ on $P_0^{\mathcal{C}} = |\Omega\rangle\langle\Omega|^{\mathcal{C}}$ and $P_1^{\mathcal{C}} = I^{\mathcal{C}} - |\Omega\rangle\langle\Omega|^{\mathcal{C}}$. $\mathcal{D}[\Phi]$ is then defined as the limit over $\{\mathcal{C}_i\}$ as $i \to \infty$, namely

$$\Pr[\mathcal{D}[\Phi] = x] := \lim_{i\to\infty} \Pr\big[\mathcal{D}^{\mathcal{C}_i}[\Phi^{\mathcal{C}_i}] = x\big],$$

which we also write $\mathcal{D}[\Phi] = \lim_{i\to\infty} \mathcal{D}^{\mathcal{C}_i}[\Phi^{\mathcal{C}_i}]$.

Note that this limit is always well-defined, because for any $i < j$,

$$\Pr\big[\mathcal{D}^{\mathcal{C}_i}[\Phi^{\mathcal{C}_i}] = 0\big] \geq \Pr\big[\mathcal{D}^{\mathcal{C}_j}[\Phi^{\mathcal{C}_j}] = 0\big]$$

and

$$\Pr\big[\mathcal{D}^{\mathcal{C}_i}[\Phi^{\mathcal{C}_i}] = 1\big] \leq \Pr\big[\mathcal{D}^{\mathcal{C}_j}[\Phi^{\mathcal{C}_j}] = 1\big],$$

but both are bounded, by 0 and 1, respectively.

The sequence of bounded cuts $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$ is essential in Definition F.2. An output at two points t and t' that have no common future—i.e., $\nexists t_0$ such that $t_0 \geq t$ and $t_0 \geq t'$—is not necessarily well-defined on both points simultaneously. This sequence then tells us which is the relevant point. Since different distinguishers might use different cuts $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$, the entire set $\mathcal{T}$ is still covered by considering the supremum over sets of distinguishers.

Definition F.3 (Limit distance)**.** Given a set of (m,n)-limit distinguishers $\mathfrak{D}$, the limit distance between two (m,n)-causal boxes Φ and Ψ is defined as

$$d^{\mathfrak{D}}(\Phi, \Psi) := \sup_{\mathcal{D}\in\mathfrak{D}} \delta(\mathcal{D}[\Phi], \mathcal{D}[\Psi]),$$

where $\delta(\cdot,\cdot)$ is the statistical or total variation distance.

As in Section 7 one can show that this distance is a pseudo-metric. It is also a metric if $\mathfrak{D}$ is the set of all distinguishers. One can also rewrite the definition using subnormalized boxes as in Appendix F.1. We do not write up these proofs, since they are nearly identical to the case of distinguishers from Definition 7.1.

Since a distinguisher $\mathcal{D}$ that takes a decision before some point $t_{\mathcal{D}}$ is a special case of a limit distinguisher with a single cut $\mathcal{C}_1 = \mathcal{T}^{\leq t_{\mathcal{D}}}$, Definition F.3 is more general than Definition 7.2. We now show that the converse also holds: we prove that for any set of limit distinguishers $\mathfrak{D}$ we can find a set of distinguishers $\mathfrak{E}$ that is at least as good at distinguishing any pair of boxes. For $i \in \mathcal{I}$ and a limit distinguisher $\mathcal{D}$ (satisfying Definition F.2), we define a distinguisher $\mathcal{D}_i$ (satisfying Definition 7.1) as follows. $\mathcal{D}_i$ behaves identically to $\mathcal{D}$ within $\mathcal{C}_i$, but ignores all inputs and produces no outputs out of $\mathcal{C}_i$, i.e., $\mathcal{D}_i$ is defined by the map $\mathcal{D}^{\mathcal{C}_i}$. The output of $\mathcal{D}_i$ is then computed on $\mathcal{T}^{\leq t_i}$ for any $t_i \in \mathcal{T}$ such that $\mathcal{C}_i \subseteq \mathcal{T}^{\leq t_i}$.

Let $\mathfrak{D}$ be any set of distinguishers, and let

$$\mathfrak{E} = \{\mathcal{D}_i : \mathcal{D} \in \mathfrak{D}, i \in \mathcal{I}\}$$

be the set obtained as described above. We now prove that $\mathfrak{E}$ is at least as good as $\mathfrak{D}$ at distinguishing causal boxes.

Lemma F.4. *Let $\mathfrak{D}$ and $\mathfrak{E}$ be defined as above. Then for any Φ and Ψ,*

$$d^{\mathfrak{D}}(\Phi, \Psi) \leq d^{\mathfrak{E}}(\Phi, \Psi).$$

Proof. For any $\mathcal{D} \in \mathfrak{D}$ one has

$$\begin{aligned}
\delta(\mathcal{D}[\Phi], \mathcal{D}[\Psi]) &= \lim_{i \to \infty} \delta(\mathcal{D}^{\mathcal{C}_i}[\Phi^{\mathcal{C}_i}], \mathcal{D}^{\mathcal{C}_i}[\Psi^{\mathcal{C}_i}]) \\
&\leq \sup_i \delta(\mathcal{D}^{\mathcal{C}_i}[\Phi^{\mathcal{C}_i}], \mathcal{D}^{\mathcal{C}_i}[\Psi^{\mathcal{C}_i}]) \\
&= \sup_i \delta(\mathcal{D}_i[\Phi], \mathcal{D}_i[\Psi]),
\end{aligned}$$

and therefore

$$\begin{aligned}
d^{\mathfrak{D}}(\Phi, \Psi) &= \sup_{\mathcal{D}} \delta(\mathcal{D}[\Phi], \mathcal{D}[\Psi]) \\
&\leq \sup_{\mathcal{D}, i} \delta(\mathcal{D}_i[\Phi], \mathcal{D}_i[\Psi]) \\
&= d^{\mathfrak{E}}(\Phi, \Psi). \qquad \square
\end{aligned}$$

Remark F.5. We get equality in Lemma F.4 if $\mathfrak{E} \subseteq \mathfrak{D}$, i.e., if for every $\mathcal{D} \in \mathfrak{D}$, the same distinguisher truncated at $\mathcal{C}_i$ is also in $\mathfrak{D}$.

G   Technical lemmas

Parallel composition of causality functions. In Proposition 6.2 we prove that the parallel composition of two causal boxes is a new valid box. To do this, we define the causality function for the new box as the union of the causality functions of both components. We prove in Lemma G.1 that this results in a valid causality function.

Lemma G.1. *Let χ_1 and χ_2 be two causality functions for some partially ordered set $\mathcal{T}$. Then the function defined as $\chi_{new}(\mathcal{C}) := \chi_1(\mathcal{C}) \cup \chi_2(\mathcal{C})$ is also a causality function.*

Proof. We need to prove that the four conditions from Definition 4.3 hold. Eq. (15) follows because

$$\begin{aligned}
\chi_{\text{new}}(\mathcal{C} \cup \mathcal{D}) &= \chi_1(\mathcal{C} \cup \mathcal{D}) \cup \chi_2(\mathcal{C} \cup \mathcal{D}) \\
&= \chi_1(\mathcal{C}) \cup \chi_1(\mathcal{D}) \cup \chi_2(\mathcal{C}) \cup \chi_2(\mathcal{D}) \\
&= \chi_{\text{new}}(\mathcal{C}) \cup \chi_{\text{new}}(\mathcal{D}).
\end{aligned}$$

Eq. (16) is satisfied because if $\mathcal{C} \subseteq \mathcal{D}$, then

$$
\begin{aligned}
\chi_{\text{new}}(\mathcal{C}) &= \chi_1(\mathcal{C}) \cup \chi_2(\mathcal{C}) \\
&\subseteq \chi_1(\mathcal{D}) \cup \chi_2(\mathcal{D}) \\
&= \chi_{\text{new}}(\mathcal{D}) \,.
\end{aligned}
$$

Since $\chi_1(\mathcal{C}) \subsetneq \mathcal{C}$ and $\chi_2(\mathcal{C}) \subsetneq \mathcal{C}$, we have that $\chi_{\text{new}}(\mathcal{C}) \subseteq \mathcal{C}$. To prove that Eq. (17) is satisfied, it remains to show that $\chi_{\text{new}}(\mathcal{C}) \neq \mathcal{C}$. This follows immediately if Eq. (18) holds for χ_{new}, since otherwise for all n, $\chi_{\text{new}}^n(\mathcal{C}) = \mathcal{C}$ and we therefore have $t \in \chi_{\text{new}}^n(\mathcal{C})$ for all n.

Hence it remains to prove that Eq. (18) is satisfied, i.e., we need to show that for every $\mathcal{C} \in \overline{\mathfrak{C}}(\mathcal{T})$ and $t \in \mathcal{C}$, there exists an $n \in \mathbb{N}$ such that $t \notin \chi_{\text{new}}^n(\mathcal{C})$. Since χ_1 and χ_2 are valid causality functions, we know that there exists an n_1 such that $t \notin \chi_1^{n_1}(\mathcal{C})$ and an n_2 such that $t \notin \chi_2^{n_2}(\mathcal{C})$. We now prove that $t \notin \chi_{\text{new}}^{n_1+n_2}(\mathcal{C})$. For $s \in \{1,2\}^n$ we define $\chi_s(\mathcal{C}) := \chi_{s_1} \circ \cdots \circ \chi_{s_n}(\mathcal{C})$. From Eq. (15) it follows that $\chi_{\text{new}}^n(\mathcal{C}) = \bigcup_{s \in \{1,2\}^n} \chi_s(\mathcal{C})$. Furthermore, combining Eqs. (16) and (17) we find that $\chi' \circ \chi \circ \chi'(\mathcal{C}) \subseteq \chi \circ \chi'(\mathcal{C}) \subseteq \chi(\mathcal{C})$. Thus, if at least k bits of s take the value 1 (2), then $\chi_s(\mathcal{C}) \subseteq \chi_1^k(\mathcal{C})$ $(\chi_s(\mathcal{C}) \subseteq \chi_1^k(\mathcal{C}))$. So $t \notin \chi_{\text{new}}^{n_1+n_2}(\mathcal{C})$. $\qquad\square$

Commuting limits. Swapping the order in which two loops are applied to a quantum box corresponds to swapping the order of the limits implicit in Eq. (31). We give here a simple condition that allows the order of the limits to be swapped, which is sufficient to prove in Lemma 6.10 that loops commute. We define the double limit as

$$
\lim_{m,n\to\infty} a_{mn} = L :\iff \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m,n \geq N, \quad |a_{mn} - L| < \varepsilon \,.
$$

Lemma G.2. *Let a_{mn} be a sequence of complex numbers indexed by $m,n \in \mathbb{N}$. If*

$$
\lim_{m\to\infty} \lim_{n\to\infty} a_{mn} \qquad\qquad and \qquad\qquad \lim_{m,n\to\infty} a_{mn}
$$

both exist, then they converge to the same value.

Proof. Let $\lim_{m,n\to\infty} a_{mn} = L$ and $\lim_{n\to\infty} a_{mn} = L_m$. We want to prove that $\lim_{m\to\infty} L_m = L$. Fix $\varepsilon > 0$ and let M be such that $\forall m,n \geq M$, $|a_{mn} - L| < \varepsilon$. For any $m \geq M$, pick $N_m \geq M$ such that $|a_{mN_m} - L_m| < \varepsilon$. Hence, there exists M such that for any $m \geq M$

$$
|L_m - L| \leq |L_m - a_{mN_m}| + |a_{mN_m} - L| < 2\varepsilon \,. \qquad\qquad\square
$$

References

[1] Gus Gutoski and John Watrous. Toward a general theory of quantum games. In *Proceedings of the 39th Symposium on Theory of Computing, STOC '07*, pages 565–574. ACM, 2007. [doi:10.1145/1250790.1250873, arXiv:quant-ph/0611234].

[2] Gus Gutoski. On a measure of distance for quantum strategies. *Journal of Mathematical Physics*, 53(3):032202, 2012. [doi:10.1063/1.3693621, arXiv:1008.4636].

[3] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Theoretical framework for quantum networks. *Physical Review A*, 80:022339, August 2009. [doi:10.1103/PhysRevA.80.022339, arXiv:0904.4483].

[4] Lucien Hardy. Reformulating and reconstructing quantum theory. eprint, 2011. [arXiv:1104.2066].

[5] Lucien Hardy. The operator tensor formulation of quantum theory. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 370(1971):3385–3417, 2012. [doi:10.1098/rsta.2011.0326, arXiv:1201.4390].

[6] Lucien Hardy. Quantum theory with bold operator tensors. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 373(2047), 2015. [doi:10.1098/rsta.2014.0239].

[7] Lucien Hardy. Probability theories with dynamic causal structure: A new framework for quantum gravity. eprint, 2005. [arXiv:gr-qc/0509120].

[8] Lucien Hardy. Towards quantum gravity: a framework for probabilistic theories with non-fixed causal structure. *Journal of Physics A: Mathematical and Theoretical*, 40(12):3081, 2007. [doi:10.1088/1751-8113/40/12/S12, arXiv:gr-qc/0608043].

[9] Ueli Maurer. Indistinguishability of random systems. In *Advances in Cryptology – EUROCRYPT 2002*, volume 2332 of *Lecture Notes in Computer Science*, pages 110–132. Springer, 2002. [doi:10.1007/3-540-46035-7_8].

[10] Ueli Maurer, Krzysztof Pietrzak, and Renato Renner. Indistinguishability amplification. In *Advances in Cryptology – CRYPTO 2007*, volume 4622 of *Lecture Notes in Computer Science*, pages 130–149. Springer, 2007. [doi:10.1007/978-3-540-74143-5_8].

[11] Vedran Dunjko, Joseph Fitzsimons, Christopher Portmann, and Renato Renner. Composable security of delegated quantum computation. In

Advances in Cryptology – ASIACRYPT 2014, Proceedings, Part II, volume 8874 of *Lecture Notes in Computer Science*, pages 406–425. Springer, 2014. [doi:10.1007/978-3-662-45608-8_22, arXiv:1301.3662].

[12] Giulio Chiribella, Giacomo Mauro D'Ariano, Paolo Perinotti, and Benoit Valiron. Quantum computations without definite causal structure. *Physical Review A*, 88:022318, August 2013. [doi:10.1103/PhysRevA.88.022318, arXiv:0912.0195].

[13] Jayne Thompson, Mile Gu, and Kavan Modi amnd Vlatko Vedral. Quantum computing with black-box subroutines. eprint, 2013. [arXiv:1310.2927].

[14] Mateus Araújo, Adrien Feix, Fabio Costa, and Časlav Brukner. Quantum circuits cannot control unknown operations. *New Journal of Physics*, 16(9):093026, 2014. [doi:10.1088/1367-2630/16/9/093026, arXiv:1309.7976].

[15] Xiao-Qi Zhou, Timothy C. Ralph, Pruet Kalasuwan, Mian Zhang, Alberto Peruzzo, Benjamin P. Lanyon, and Jeremy L. O'Brien. Adding control to arbitrary unknown quantum operations. *Nature Communications*, 2:413, August 2011. [doi:10.1038/ncomms1392, arXiv:1006.2670].

[16] Nicolai Friis, Vedran Dunjko, Wolfgang Dür, and Hans J. Briegel. Implementing quantum control for unknown subroutines. *Physical Review A*, 89:030303, March 2014. [doi:10.1103/PhysRevA.89.030303, arXiv:1401.8128].

[17] Mateus Araújo, Fabio Costa, and Časlav Brukner. Computational advantage from quantum-controlled ordering of gates. *Physical Review Letters*, 113:250402, December 2014. [doi:10.1103/PhysRevLett.113.250402, arXiv:1401.8127].

[18] Lorenzo M. Procopio, Amir Moqanaki, Mateus Araújo, Fabio Costa, Irati Alonso Calafell, Emma G. Dowd, Deny R. Hamel, Lee A. Rozema, Časlav Brukner, and Philip Walther. Experimental superposition of orders of quantum gates. *Nature Communications*, 6:7913, August 2015. [doi:10.1038/ncomms8913, arXiv:1412.4006].

[19] Timoteo Colnaghi, Giacomo Mauro D'Ariano, Stefano Facchini, and Paolo Perinotti. Quantum computation with programmable connections between gates. *Physics Letters A*, 376(45):2940 – 2943, 2012. [doi:10.1016/j.physleta.2012.08.028, arXiv:1109.5987].

[20] Giulio Chiribella. Perfect discrimination of no-signalling channels via quantum superposition of causal structures. *Physical Review A*, 86:040301, October 2012. [doi:10.1103/PhysRevA.86.040301, arXiv:1109.5154].

[21] Adrien Feix, Mateus Araújo, and Časlav Brukner. Quantum superposition of the order of parties as a communication resource. *Physical Review A*, 92:052326, November 2015. [doi:10.1103/PhysRevA.92.052326, arXiv:1508.07840].

[22] Dominique Unruh. Simulatable security for quantum protocols. eprint, 2004. [arXiv:quant-ph/0409125].

[23] Dominique Unruh. Universally composable quantum multi-party computation. In *Advances in Cryptology – EUROCRYPT 2010*, volume 6110 of *Lecture Notes in Computer Science*, pages 486–505. Springer, 2010. [doi:10.1007/978-3-642-13190-5_25, arXiv:0910.2912].

[24] Michael Ben-Or and Dominic Mayers. General security definition and composability for quantum & classical protocols. eprint, 2004. [arXiv:quant-ph/0409062].

[25] Ognyan Oreshkov and Christina Giarmatzi. Causal and causally separable processes. *New Journal of Physics*, 18(9):093020, 2016. [doi:10.1088/1367-2630/18/9/093020, arXiv:1506.05449].

[26] Mateus Araújo, Cyril Branciard, Fabio Costa, Adrien Feix, Christina Giarmatzi, and Časlav Brukner. Witnessing causal nonseparability. *New Journal of Physics*, 17(10):102001, 2015. [doi:10.1088/1367-2630/17/10/102001, arXiv:1506.03776].

[27] Ueli Maurer and Renato Renner. Abstract cryptography. In *Proceedings of Innovations in Computer Science, ICS 2011*, pages 1–21. Tsinghua University Press, 2011.

[28] Christopher Portmann and Renato Renner. Cryptographic security of quantum key distribution. eprint, 2014. [arXiv:1409.3525].

[29] Ueli Maurer. Authentication amplification by synchronization. In *Proceedings of the 2013 IEEE International Symposium on Information Theory, ISIT 2013*, pages 2711–2714. IEEE, 2013. [doi:10.1109/ISIT.2013.6620719].

[30] Christopher Portmann. Key recycling in authentication. *IEEE Transactions on Information Theory*, 60(7):4383–4396, July 2014. [doi:10.1109/TIT.2014.2317312, arXiv:1202.1229].

[31] Adrian Kent. Unconditionally secure bit commitment. *Physical Review Letters*, 83:1447–1450, August 1999. [doi:10.1103/PhysRevLett.83.1447, arXiv:quant-ph/9810068].

[32] Adrian Kent. Unconditionally secure bit commitment by transmitting measurement outcomes. *Physical Review Letters*, 109:130501, September 2012. [doi:10.1103/PhysRevLett.109.130501, arXiv:1108.2879].

[33] Jędrzej Kaniewski, Marco Tomamichel, Esther Hänggi, and Stephanie Wehner. Secure bit commitment from relativistic constraints. *IEEE Transactions on Information Theory*, 59(7):4687–4699, July 2013. [doi:10.1109/TIT.2013.2247463, arXiv:1206.1740].

[34] T. Lunghi, J. Kaniewski, F. Bussières, R. Houlmann, M. Tomamichel, S. Wehner, and H. Zbinden. Practical relativistic bit commitment. *Physical Review Letters*, 115:030502, July 2015. [doi:10.1103/PhysRevLett.115.030502, arXiv:1411.4917].

[35] Emily Adlam and Adrian Kent. Deterministic relativistic quantum bit commitment. *International Journal of Quantum Information*, 13(05):1550029, 2015. [doi:10.1142/S021974991550029X, arXiv:1504.00943].

[36] Harry Buhrman, Nishanth Chandran, Serge Fehr, Ran Gelles, Vipul Goyal, Rafail Ostrovsky, and Christian Schaffner. Position-based quantum cryptography: Impossibility and constructions. In *Advances in Cryptology – CRYPTO 2011*, volume 6841 of *Lecture Notes in Computer Science*, pages 429–446. Springer, 2011. [doi:10.1007/978-3-642-22792-9_24, arXiv:1009.2490, IACR e-print: 2010/275].

[37] Dominique Unruh. Quantum position verification in the random oracle model. In *Advances in Cryptology – CRYPTO 2014*, volume 8617 of *Lecture Notes in Computer Science*, pages 1–18. Springer, 2014. [doi:10.1007/978-3-662-44381-1_1, IACR e-print: 2014/118].

[38] Dominique Unruh and Jörn Müller-Quade. Universally composable incoercibility. In *Advances in Cryptology – CRYPTO 2010*, volume 6223 of *Lecture Notes in Computer Science*, pages 411–428. Springer, 2010. [doi:10.1007/978-3-642-14623-7_22, IACR e-print: 2009/520].

[39] Christian Matt and Ueli Maurer. The one-time pad revisited. In *Proceedings of the 2013 IEEE International Symposium on Information Theory, ISIT 2013*, pages 2706–2710. IEEE, 2013. [doi:10.1109/ISIT.2013.6620718].

[40] Lucien Hardy. Quantum gravity computers: On the theory of computation with indefinite causal structure. In Wayne C. Myrvold and Joy Christian, editors, *Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle: Essays in Honour of Abner Shimony*, pages 379–401. Springer, 2009. [doi:10.1007/978-1-4020-9107-0_21, arXiv:quant-ph/0701019].

[41] Lucien Hardy. Formalism locality in quantum theory and quantum gravity. In Alisa Bokulich and Gregg Jaeger, editors, *Philosophy of Quantum*

Information and Entanglement, pages 44–62. Cambridge University Press, 2010. [arXiv:0804.0054].

[42] Ognyan Oreshkov, Fabio Costa, and Časlav Brukner. Quantum correlations with no causal order. *Nature Communications*, 3:1092, October 2012. [doi:10.1038/ncomms2076, arXiv:1105.4464].

[43] Časlav Brukner. Quantum causality. *Nature Physics*, 10(4):259–263, April 2014. [doi:10.1038/nphys2930].

[44] Cyril Branciard. Witnesses of causal nonseparability: an introduction and a few case studies. *Scientific Reports*, 6:26018, May 2016. [doi:10.1038/srep26018, arXiv:1603.00043].

[45] Ämin Baumeler and Stefan Wolf. Perfect signaling among three parties violating predefined causal order. In *Proceedings of the 2014 IEEE International Symposium on Information Theory, ISIT 2014*, pages 526–530. IEEE, 2014. [doi:10.1109/ISIT.2014.6874888, arXiv:1312.5916].

[46] Časlav Brukner. Bounding quantum correlations with indefinite causal order. *New Journal of Physics*, 17(8):083034, 2015. [doi:10.1088/1367-2630/17/8/083034, arXiv:1404.0721].

[47] Cyril Branciard, Mateus Araújo, Adrien Feix, Fabio Costa, and Časlav Brukner. The simplest causal inequalities and their violation. *New Journal of Physics*, 18(1):013008, 2016. [doi:10.1088/1367-2630/18/1/013008, arXiv:1508.01704].

[48] Ämin Baumeler and Stefan Wolf. Device-independent test of causal order and relations to fixed-points. *New Journal of Physics*, 18(3):035014, 2016. [doi:10.1088/1367-2630/18/3/035014, arXiv:1511.05444].

[49] Ämin Baumeler, Adrien Feix, and Stefan Wolf. Maximal incompatibility of locally classical behavior and global causal order in multiparty scenarios. *Physical Review A*, 90:042106, October 2014. [doi:10.1103/PhysRevA.90.042106, arXiv:1403.7333].

[50] Ämin Baumeler and Stefan Wolf. The space of logically consistent classical processes without causal order. *New Journal of Physics*, 18(1):013036, 2016. [doi:10.1088/1367-2630/18/1/013036, arXiv:1507.01714].

[51] Ämin Baumeler and Stefan Wolf. Non-causal computation avoiding the grandfather and information antinomies. eprint, 2016. [arXiv:1601.06522].

[52] Ognyan Oreshkov and Nicolas J. Cerf. Operational quantum theory without predefined time. *New Journal of Physics*, 18(7):073037, 2016. [doi:10.1088/1367-2630/18/7/073037, arXiv:1406.3829].

[53] Christian Matt, Ueli Maurer, Christopher Portmann, Renato Renner, and Björn Tackmann. Toward an algebraic theory of systems. eprint, 2016. [arXiv:1609.04293].

[54] Lucien Hardy. On the theory of composition in physics. In Bob Coecke, Luke Ong, and Prakash Panangaden, editors, *Computation, Logic, Games, and Quantum Foundations. The Many Facets of Samson Abramsky: Essays Dedicated to Samson Abramsky on the Occasion of His 60th Birthday*, pages 83–106. Springer, 2013. [doi:10.1007/978-3-642-38164-5_7, arXiv:1303.1537].

[55] Edward A. Lee and Alberto Sangiovanni-Vincentelli. A framework for comparing models of computation. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 17(12):1217–1229, December 1998. [doi:10.1109/43.736561].

[56] Vern Paulsen. *Completely Bounded Maps and Operator Algebras*. Number 78 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2003. [doi:10.1017/CBO9780511546631].

[57] John Watrous. Theory of quantum information, 2016. Book draft, https://cs.uwaterloo.ca/~watrous/TQI/.

[58] Alexander S. Holevo. The Choi-Jamiołkowski forms of quantum Gaussian channels. *Journal of Mathematical Physics*, 52(4):042202, 2011. [doi:10.1063/1.3581879, arXiv:1004.0196].

[59] Paul Benioff. The computer as a physical system: A microscopic quantum mechanical hamiltonian model of computers as represented by turing machines. *Journal of Statistical Physics*, 22(5):563–591, 1980. [doi:10.1007/BF01011339].

[60] David Deutsch. Quantum theory, the church-turing principle and the universal quantum computer. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 400(1818):97–117, 1985. [doi:10.1098/rspa.1985.0070].

[61] Andrew Chi-Chih Yao. Quantum circuit complexity. In *Proceedings of the 34st Symposium on Foundations of Computer Science, FOCS '93*, pages 352–361. IEEE, 1993. [doi:10.1109/SFCS.1993.366852].

[62] Ethan Bernstein and Umesh Vazirani. Quantum complexity theory. *SIAM Journal on Computing*, 26(5):1411–1473, October 1997. [doi:10.1137/S0097539796300921].

[63] David Deutsch. Quantum computational networks. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 425(1868):73–90, 1989. [doi:10.1098/rspa.1989.0099].

[64] John Goold, Marcus Huber, Arnau Riera, Lídia del Rio, and Paul Skrzypczyk. The role of quantum information in thermodynamics—a topical review. *Journal of Physics A: Mathematical and Theoretical*, 49(14):143001, 2016. [doi:10.1088/1751-8113/49/14/143001, arXiv:1505.07835].

[65] Philippe Faist. *Quantum Coarse-Graining: An Information-Theoretic Approach to Thermodynamics*. PhD thesis, Swiss Federal Institute of Technology (ETH) Zurich, 2016. [arXiv:1607.03104].

[66] Adrien Feix, Mateus Araújo, and Časlav Brukner. Causally non-separable processes admitting a causal model. *New Journal of Physics*, 18(8):083040, 2016. [doi:10.1088/1367-2630/18/8/083040, arXiv:1604.03391].

[67] Bernd S. W. Schröder. *Ordered Sets: An Introduction*. Birkhäuser Boston, 2003. [doi:10.1007/978-1-4612-0053-6].