Signature Schemes Based on the Strong RSA Assumption

Ronald Cramer
Institute for Theoretical Computer Science, ETH Zurich, 8092 Zurich, Switzerland
cramer@inf.ethz.ch

Victor Shoup
IBM Zurich Research Laboratory, Säumerstr. 4, 8803 Rüschlikon, Switzerland
sho@zurich.ibm.com

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Abstract

We describe and analyze a new digital signature scheme. The new scheme is quite efficient, does not require the the signer to maintain any state, and can be proven secure against adaptive chosen message attack under a reasonable intractability assumption, the so-called Strong RSA Assumption. Moreover, a hash function can be incorporated into the scheme in such a way that it is also secure in the random oracle model under the standard RSA Assumption.

1 Introduction

We describe new, efficient digital signature schemes whose security is based on the Strong RSA Assumption.

By security, we mean security against an adaptive chosen message attack, as defined in [9]. To prove that our new schemes are secure, we need to make the Strong RSA Assumption (SRA), recently introduced by [2]. We also need a collision-resistant hash function.[1]

Our new schemes are interesting in that they are state-free, unlike other provably secure schemes [9, 7, 6]. Of course, we achieve this at the expense using of a potentially stronger assumption than is made in [9, 7, 6].

[1] We could get by using a target collision-resistant hash function by adapting the ideas from [4].

We stress that in discussing proofs of security, we *are not* making use of the "random oracle" model of computation [3], but rather, we are working in the "real world" of computation. Indeed, the standard "hash and invert" RSA signature is provably secure in the random oracle model under the standard RSA Assumption.

We also make the further observation that another hash function can be incorporated into our new schemes in such a way that they are also secure in the random oracle model under the standard RSA Assumption. In this sense, our schemes can be made to be at least as secure as a standard RSA signature.

The SRA is the assumption that the following problem is hard to solve. Given a randomly chosen RSA modulus n and a random $z \in \mathbf{Z}_n^*$, find $r > 1$ and $y \in \mathbf{Z}_n^*$ such that $y^r = z$. Note that this differs from the ordinary RSA Assumption (RA), in that for RA, the exponent r is chosen independently of z, whereas for SRA, r may be chosen in a way that depends on z. The SRA is a potentially stronger assumption than the RA, but at the present time, the only known method for breaking either RA or SRA is to solve the integer factorization problem.

Independently, Gennaro, Halevi, and Rabin [8] have also recently discovered efficient, state-free signature schemes based on the SRA. Our schemes are actually quite different from theirs, and we think that all of these different schemes are of interest from both a theoretical and practical perspective, because they are the only truly practical and state-free schemes available that admit a proof of security under a natural intractability assumption. Moreover, our scheme is potentially more efficient for the following reason. The paper [8] contains several signature schemes, but the only fully proved scheme requires a "trapdoor" or "chameleon" collision-resistant hash function with the following very special property: its output is a prime number. Implementing such a hash function is both awkward and potentially computationally expensive. Indeed, depending on the security parameters and implementation details, evaluating this hash function can dominate the running time of the signing algorithm. Our scheme sidesteps this problem altogether. While the signing algorithm still has to generate a prime number, it has a great deal of flexibility in how this is done, yielding a much more efficient algorithm. Basically, our signing algorithm just needs to generate *any* prime number of appropriate length (e.g., 161-bits) subject only to the requirement that the probability of generating the same prime twice is small.

Our new schemes can be seen as variations of the scheme of Cramer and Damgard [6], which itself can be seen as an adaptation of the identification

scheme of Guillou and Quisquater [10]. In §2, we present and analyze our basic scheme. In §3, we present and analyze a variation based on trapdoor hashing. In §4, we sketch an algorithm for fast prime generation, as required by the signing algorithm, and discuss how the intractability assumption can be weakened by using hash functions.

2 The Basic Scheme

In this section we describe the basic scheme, and give a proof of its security.

The scheme is parameterized by two security parameters, k and l, where $l + 1 < k$. Reasonable choices might be $k = 512$ and $l = 160$. The scheme makes use of a collision-resistant hash function H whose output can be interpreted as a positive integer less than 2^l. A reasonable choice for H might be SHA-1.

For a positive integer n, we let QR_n denote the subgroup of $\mathbf{Z}_n^*$ of squares (i.e., the quadratic residues modulo n).

Key Generation Two random k-bit primes p and q are chosen, where $p = 2p' + 1$ and $q = 2q' + 1$, with both p' and q' prime. Let $n = pq$. Also chosen are:

- random $h, x \in \mathrm{QR}_n$;
- a random $(l + 1)$-bit prime e'.

The public key is

$$(n, h, x, e').$$

The private key is

$$(p, q).$$

Signature Generation To sign a message m (an arbitrary bit string), a random $(l + 1)$ bit prime $e \neq e'$ is chosen, and a random $y' \in \mathrm{QR}_n$ is chosen. The equation

$$y^e = xh^{H(x')}$$

is solved for y, where x' satisfies the equation

$$(y')^{e'} = x'h^{H(m)}.$$

Note that y can be calculated using the factorization of n in the private key. The signature is

$$(e, y, y').$$

Signature Verification To verify a putative signature (e, y, y') on a message m, it is first checked that e is an odd $(l+1)$-bit number different from e'. Second, $x' = (y')^{e'} h^{-H(m)}$ is computed. Third, it is checked that $x = y^e h^{-H(x')}$.

Implementation Notes

We remark that the signature verification algorithm *does not* need to verify that e is prime.

To speed both verification and signing, the public key might contain h^{-1} instead of h.

In generating a signature, the only full-length exponentiation that needs to be performed is in the computation of y. The cost of this can be significantly reduced, as follows. First, we can arrange that $x = h^a$ for a random number $a \bmod p'q'$, where a is stored in the secret key. This is acceptable, because h is with overwhelming probability a generator of QR_n, and thus the distribution of the public key does not change significantly. Now, if d is the inverse of $e \bmod p'q'$, then $y = h^b$, where $b = da + dH(x') \bmod p'q'$. So the computation of y involves exponentiation with the *fixed* base h to the power b. Using pre-computation techniques [13], we can substantially reduce the number of modular multiplications using a table of pre-computed numbers.

Of course, in all of the above, one utilizes the Chinese Remainder Theorem as well to speed the exponentiations.

We also note that the primes generated by the signer do not have to be random primes. The only requirement is that the probability of generating the same prime twice is negligible.

Using these implementation ideas, together with a fast prime generator like the one described in §4, it would appear that the cost of signing is almost the same as that of basic RSA. Verification, however, will still be slower, involving basically two 160-bit exponentiations.

Proof of Security

Now we proceed to prove the security of the above scheme.

Theorem 1 *The above signature scheme is secure against adaptive chosen message attack, assuming the SRA and assuming that H is collision-resistant.*

To prove this theorem, let us consider a forging algorithm that makes t signing queries and then produces a forgery. For $1 \leq i \leq t$, let m_i be the ith message signed, let (e_i, y_i, y_i') be the i signature, and let x_i' be defined as $x_i' = (y_i')^{e'} h^{-H(m_i)}$. Let (e, y, y') be the forgery on message m (so $m \neq m_i$ for all $1 \leq i \leq t$). Also, let $x' = (y')^{e'} h^{-H(m)}$.

We distinguish between three types of forgeries:

Type I For some $1 \leq j \leq t$, $e = e_j$ and $x' = x_j'$.

Type II For some $1 \leq j \leq t$, $e = e_j$ and $x' \neq x_j'$.

Type III For all $1 \leq i \leq t$, $e \neq e_i$.

Assuming that the probability of generating two equal e_i values is negligible, a forgery has a unique type.

If there is a forger that succeeds with non-negligible probability, then there exists either Type I forger, a Type II forger, or a Type III forger, one of which succeeds with non-negligible probability. We show that any of these forgers can be turned into an algorithm breaking the SRA. In fact, a forger of Types I or II can be used to break the RA, and the proof of this is quite similar to proofs in [6]. We only need the SRA in case the forger is Type III.

Type I Forger

Suppose we have a Type I forger that succeeds with non-negligible probability. We want to show how to use this forger to break the RA. That is, we are given n, a random $z \in \mathbf{Z}_n^*$, and a random $(l+1)$-bit prime r, and we want to compute $z^{1/r}$.

We describe a simulator that interacts with the forger. We choose random $(l+1)$-bit primes $e_1, \ldots, e_t$, and we create a public key as follows. We set

$$h = z^{2 \prod_i e_i}.$$

We next choose $w \in \mathbf{Z}_n^*$ at random and set

$$x = w^{2 \prod_i e_i}.$$

Finally, we set $e' = r$.

Now, to sign message m_i, the simulator chooses $y_i' \in \mathrm{QR}_n$ at random, and computes $x_i' = (y_i')^{e'} h^{-H(m_i)}$. Next, the simulator solves the equation $y_i^e = x h^{H(x_i')}$ for y_i, which is can easily do, since it knows the e_ith roots of x and h.

It is easy to see that the simulator perfectly simulates the forger's view.

Now suppose the forger creates a Type I forgery (e, y, y') on a message m. So for some $1 \leq j \leq t$, $e = e_j$ and $x' = x'_j$. This yields two equations

$$\begin{aligned} (y')^{e'} &= x' h^{H(m)}; \\ (y'_j)^{e'} &= x' h^{H(m_j)}. \end{aligned}$$

Since we are assuming H is collision-resistant, we may assume that $H(m) \neq H(m_j)$. Thus, dividing these two equations, we can calculate $v \in \mathbf{Z}_n^*$ and an integer $a \not\equiv 0 \pmod{e'}$ such that

$$v^{e'} = h^a = z^{2a \prod_i e_i}.$$

Moreover, since $\gcd(2a \prod_i e_i, e') = 1$ and $e' = r$, we can easily compute an rth root of z. This is done via a standard procedure, which we will need to use several times, and we recall it for completeness. If we have $v^r = z^b$, where $\gcd(r, b) = 1$, then we compute b' such that $bb' = 1 + rk$. It follows that $(v^{b'} z^{-k})^r = z$.

Type II Forger

As in the Type I case, we are given n, $z \in \mathbf{Z}_n^*$ and r, and we want to find an rth root of z.

We may assume that the value j in the definition of a Type II forgery is fixed. If not, we can guess it.

Again, we describe a simulator. We create a public key as follows. For $1 \leq i \leq t$, with $i \neq j$, we choose e_i to be a random $(l+1)$-bit prime. We set $e_j = r$. We also select e' to be a random $(l+1)$-bit prime. We set

$$h = z^{2e' \prod_{i \neq j} e_i}.$$

We choose $w \in \mathbf{Z}_n^*$ at random, and set

$$y_j = w^{2 \prod_{i \neq j} e_i}.$$

We choose $u \in \mathbf{Z}_n^*$ at random, and set

$$x'_j = e^{2e'}.$$

We compute

$$x = y_j^{e_j} h^{-H(x'_j)}.$$

Next, we describe how to sign message m_i. First, suppose $i \neq j$. We choose $y'_i \in \mathrm{QR}_n$ at random, and compute as $x'_i = (y'_i)^{e'} h^{-H(m_i)}$. Then, since we know the e_ith roots of x and h, we can easily compute the corresponding value y_i.

Second, suppose $i = j$. Since we know the e'th roots of h and x'_j, we can compute the correct value y'_j. The correct value of y_j has already been determined.

That completes the description of the simulator. It is easy to see that the simulator perfectly simulates the forger's view.

Now suppose the forger creates a Type II forgery (e, y, y') on a message m, where $e = e_j$ and $x' \neq x'_j$. Then we have

$$\begin{aligned} y^e &= xh^{H(x')}; \\ y_j^e &= xh^{H(x'_j)}. \end{aligned}$$

Then by an argument similar to that in the Type I case, we can divide these two equations, and calculate an rth root of z.

Type III Forger

Given a Type III forger, we show how to break the SRA. That is, given n and $z \in \mathbf{Z}_n^*$, compute $r > 1$ and an rth root of z.

The simulator runs as follows. We choose random $(l+1)$-bit primes $e', e_1, \ldots, e_t$. We set

$$h = z^{2e' \prod_i e_i}.$$

Now we choose a random $a \in \{1, \ldots, n^2\}$, and set $x = h^a$.

Now, by construction, QR_n is a cyclic group of order $p'q'$. We can assume that h generates QR_n, since this happens with overwhelming probability.

Now let $a = bp'q' + c$, where $0 \leq c < p'q'$. Because a was chosen at random from a suitably large interval, the distribution of c is statistically indistinguishable the uniform distribution on $\{0, \ldots, p'q' - 1\}$. Moreover, the conditional distribution of b given c is statistically indistinguishable from the uniform distribution on $\{0, \ldots, \lfloor n^2/p'q' \rfloor\}$. That is, c and b are essentially independent.

Because the distribution of c is essentially uniform, x is essentially distributed like a random element of QR_n. Since we know all the relevant roots of x and h, we can easily sign all messages.

Now suppose the forger creates a Type III forgery, (e, y, y'). Then we have

$$y^e = xh^{H(x')} = z^m,$$

where

$$m = 2e' \prod_i e_i \cdot (a + H(x')).$$

Let $d = \gcd(e, m)$. Now, using the same procedure as was used in the Type I and Type II cases, we can compute an $(e/\gcd(e, m))$-th root of z, which is nontrivial provided $e \not| m$. So it suffices to show that $e \not| m$ with non-negligible probability. Let r be a prime dividing e. Now, $r \not| 2e' \prod_i e_i$ by construction. So it suffices to show that $r \not| (a + H(x'))$ with non-negligible probability. Let $a = bp'q' + c$ as above. Now, r may depend on c, but we observed above that c and b are essentially independent. And since by construction $r \not| p'q'$, it follows that $r \mid (a + H(x'))$ with probability very close to $1/r$, as we are evaluating a linear polynomial in b at a random point. Thus, with non-negligible probability, $r \not| (a + H(x'))$.

3 Trapdoor Hash Scheme

Consider the basic signature scheme presented above, and consider a signature (e, y, y') on a message m. Let $x' = (y')^{e'} h^{-H(m)}$. Then we have $y^e = xh^{H(x')}$.

One can view the value $H(x')$ as a kind of "trapdoor hash," also called a "chameleon hash" (see [12] for detailed discussion, references, and further applications). One can also base a trapdoor hash on the Discrete Logarithm Assumption in a standard way, as follows. Let g_1, g_2 be two random generators for a group G of order s, where s is an $(l + 1)$-bit prime. To hash a message m, we compute the hash value $\alpha = H(g_1^t g_2^{H(m)})$, where H is an ordinary, collision-resistant hash function, and t is chosen at random mod s. In addition to the hash value α, we also output the side information t. The trapdoor in this scheme is the g_1 logarithm of g_2. A simulator that knows the trapdoor can construct a hash value α without knowing m, and then later, given m, can construct and the appropriate side information t.

We now describe a signature scheme based on this.

Key Generation Two random k-bit primes p and q are chosen, where $p = 2p' + 1$ and $q = 2q' + 1$, with both p' and q' prime. Let $n = pq$. Also chosen are:

- random $h, x \in \mathrm{QR}_n$;
- a group G of order s, where s is an $(l+1)$-bit prime, and two random generators g_1, g_2 of G.

The public key is

$$(n, h, x, g_1, g_2),$$

along with an appropriate description of G (including s). The private key is

$$(p, q).$$

Signature Generation To sign a message m (an arbitrary bit string), a random $(l+1)$ bit prime e is chosen, and a random $t \in \mathbf{Z}_s$ is chosen. The equation

$$y^e = x h^{H(g_1^t g_2^{H(m)})}$$

is solved for y. The signature is

$$(e, y, t).$$

Signature Verification To verify a putative signature (e, y, t) on a message m, it is first checked that e is an odd $(l+1)$-bit number. Second, it is checked that

$$x = y^e h^{-H(g_1^t g_2^{H(m)})}.$$

Theorem 2 *The above signature scheme is secure against adaptive chosen message attack, assuming the SRA, assuming that H is collision-resistant, and assuming the Discrete Logarithm Assumption for the group G.*

The proof of this theorem is very similar to the proof of Theorem 1. We leave the details to the reader.

In a variation on this scheme, we give the signing algorithm the trapdoor to the hash. The advantage of doing this can be appreciated if one makes a distinction between the "off line" and "on line" cost of signing. If the signer has the trap door, then in fact the "on line" cost is essentially a single multiplication mod s—all of the other work in creating the signature can be done before the message m is actually received.

4 Remarks on Prime Generation

In our signature scheme, the signer must generate a random $(l+1)$ bit prime with each signature. As we remarked already, these primes need not be chosen from the uniform distribution $(l+1)$-bit primes. The only requirement is that the probability of generating two equal primes should be negligible. Thus, we have quite a bit of flexibility in how we generate these primes. This is perhaps important, because if one is not careful, the cost of prime generation can easily be the dominant cost of signing. This is especially so if one wants a completely rigorous algorithm with a sufficiently small error probability.

For example, suppose one uses the Miller-Rabin test [16] to test for primality. Suppose $l = 160$. Further, suppose we want an error rate of 2^{-96}, which will allow us to make 2^{32} signatures with an overall error rate of 2^{-64}. Now suppose we choose random 161-bit numbers until we have found a number that passes a number of trial divisions and a single Miller-Rabin test Along the way, we will make a small handful of Miller-Rabin tests that reject some composite numbers that pass the trial division test. Then we will need to perform about 47 additional Miller-Rabin tests to achieve the desired error rate. Working with a 1024-bit RSA modulus, empirical tests suggest that the cost of all these Miller-Rabin tests is much more than the cost of all the other steps in the signing algorithm. Moreover, the best results we know of [11] on the error rate of the Miller-Rabin test do not improve this situation much.

A Fast Prime Generation Algorithm

Here we sketch a very efficient algorithm for generating primes as required by the signing algorithm. Again, assume $l = 160$. So we need to generate a 161-bit prime. To do this, we first generate a random prime P in the range $(2^{52}, 2^{53})$. Because P is small enough, the primality of P can be quickly verified using one of a number of procedures described in Bleichenbacher's thesis [5, Chapter 3], which are correct for primes up to $10^{16} > 2^{53}$. For example, one of Bleichenbacher's results, as reported in [14], states that the Miller-Rabin test for the bases $2, 3, 5, 7, 11, 13$ and 23 is a correct primality test for numbers in this range. Next, we repeatedly choose integers R in the interval $((2^{160}-1)/8P, (2^{161}-1)/8P)$ until $e = 8RP+1$ is prime. Lemma 2 in [14] provides an extremely efficient probabilistic algorithm for generating a certificate of primality for numbers of this form, requiring essentially just a single exponentiation on average to certify a prime; our choice of parameters

ensures that $4P \geq e^{1/3}$, as required by that lemma.

Lemma 1 *With this procedure, the expected number of trials until P is prime is at most 64. Assuming the Generalized Riemann Hypothesis, the following holds. For any fixed P, the number of trials until $8RP + 1$ is prime is at most 128. The probability that two independent runs of this procedure output the same prime is at most 2^{-141}.*

To prove this we use some explicit estimates from [1]. First, by Theorem 8.8.1 in [1], we have the number of primes P in the given range is more than 2^{46}. The first claim in the lemma follows trivially.

For the second and third claims, we use Theorem 8.8.18 in [1], which gives a very sharp estimate on the number of primes in an arithmetic progression, assuming the Generalized Riemann Hypothesis. Using a simple calculation, this theorem implies that for any fixed P, there are at least 2^{98} primes of the form $8RP + 1$ in the range $(2^{160}, 2^{161})$. The second claim in the lemma is now follows easily.

Now for the third claim. Let $e_i = 8P_iR_i + 1$ for $i = 1, 2$ be two primes generated by independent executions of the algorithm. We have

$$\Pr[e_1 = e_2] \leq \Pr[e_1 = e_2 | P_1 = P_2] \Pr[P_1 = P_2] + \Pr[e_1 = e_2 | P_1 \neq P_2] \quad (1)$$

By the previous observations, the first term in (1) is bounded by $2^{-98} \cdot 2^{-46} = 2^{-144}$. Now fix $P_1 \neq P_2$. The second term in (1) is bounded by the probability that the following two events occur: $P_1 \mid R_2$ and $R_1 = (R_2/P_1)P_2$. By a simple calculation, the first event occurs with probability at most 2^{-44} and the second with probability at most 2^{-98}. As these events are independent, we get a bound of 2^{-142} for the second term in (1). This implies the lemma.

We do not claim that this is the best way to generate 161-bit primes, or even particularly original, but it seems like a reasonable one, and it is certainly much more efficient than the cost of iterating the Miller-Rabin test. In [15] a similar approach to generating certified primes is presented, but their analysis is insufficient for our purposes, as it is only asymptotic—we need explicit bounds, and thus have to appeal to the Generalized Riemann Hypothesis. Moreover, their approach would anyway give a higher collision probability.

Since the technique suggested here provides a certificate of primality that is small and easily verified, one could augment the signature scheme by adding this certificate to the signature and having the verifier check it.

This can only improve security, allowing us to weaken the SRA so that the adversary's exponent has to be a certified prime of the proper form, and not just an arbitrary integer.

Using a Hash Function

We can weaken the intractability assumption even further. Suppose that in the above algorithm for generating a prime, we require that the random numbers P and R are outputs of a cryptographic hash function. The signing algorithm can feed random bits into such a hash function, and if the hash function is nearly uniform, the same properties that are proved above will still hold.

The hash function inputs that yield P and R (respectively) are included as part of the signature, and the verifier checks that these values are correct. By doing this, we greatly constrain the adversary's attack strategy, allowing us to weaken the SRA so that the adversary's exponent is a certified prime of this very special form. This intuitively seems like a much harder problem, and indeed, this intuition is somewhat justified by the fact that is is straightforward to prove *in the random oracle model* that the resulting signature scheme is secure under a *standard* RSA assumption.

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